Simultaneous Binary Collisions for Collinear Four-Body Problem

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Abstract

In this paper, we use canonical transformations to collectively analytically continue the singularities of the simultaneous binary collision solutions for the collinear four-body problem in both the decoupled case and the coupled case. And more importantly, we describe the relationship between the decoupled solutions and the coupled solutions.

1 Introduction

The question of the regularization for a simultaneous binary collision (SBC) solution is not completely understood although many results about it have been obtained. In [1] , Saari showed that a SBC solution can be analytic continued by rescaling the time $s = t^{\frac{1}{3}}$ and a majorant method argument. In [3], Simo and Lacomba gave a different approach and also they showed that "simultaneous binary collisions in the classical n-body problem are C^0 block-regularizable". In [2], Elbialy proved that "collision and ejection orbits can be collectively analytically continued, i.e. each collision-ejection orbit can be written as a convergent power series in $t^{\frac{1}{3}}$, with coefficients that depend real analytically on the initial conditions". In [4], Martinez and Simo also discussed the block regularization and the result is this "regularization is differentiable but the map passing from initial to final conditions is exactly $C^{8/3}$". In [5], Punosevac and Wang constructed coordinate transforms that removed the singularities of simultaneous binary collisions in a pair of decoupled Kepler problems and in a restricted collinear four-body problem.

In this paper, we use similar canonical transformations and time transform $s = s(t)$, which were used by Siegel and Moser([6]) to do the analytic branch regularization at the singular point for a binary collision.To the best of our knowledge, this is the first time canonical transformations are introduced for SBC. In Section 3, we use the extended analytic implicit function theorem to show the branch regularization for the decoupled case with special masses. And also, we find the special constant C, which is crucial for the decoupled case. In Section 4, we use the majorant method argument to prove the branch regularization for the coupled case with special masses. In Section 6, the coupled case with general masses are

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discussed and it follows the same argument as Section 4. In Section 5, we calculate the Taylor series solutions for both the decoupled case and the coupled case. By comparing them, lots of connections are given.

Express the equations of motion for the collinear four-body problem in Hamiltonian form, assuming that the center of mass P_0 rests at origin. We denote the coordinates of the points P_1, P_2, P_3, P_4 by q_1, q_2, q_3, q_4, and denote the momenta by p_1, p_2, p_3, p_4 respectively, where $p_k = m_k\dot{q}_k$ (k=1, 2, 3, 4). Without loss of generality, we assume $q_1 < q_2 < q_3 < q_4$ on the line, $p_1 > 0$, $p_3 > 0$, $p_2 < 0$ and $p_4 < 0$. For the collinear four-body problem, we have the Hamiltonian system:

$$\begin{cases} \dot{q}_k = E_{p_k} \\ \dot{p}_k = -E_{q_k} \end{cases}, \quad E = T - U(k = 1, 2, 3, 4)$$

$$T = \frac{1}{2}\sum_{k=1}^{4}\frac{p_k^2}{m_k},$$

$$U = \sum_{1\leq j\leq i\leq 4}\frac{m_i m_j}{\mid q_i - q_j \mid}.$$

Also, assume the center of mass and the total momentum both are o. i.e.

$$m_1q_1 + m_2q_2 + m_3q_3 + m_4q_4 = 0,$$

$$m_1\dot{q}_1 + m_2\dot{q}_2 + m_3\dot{q}_3 + m_4\dot{q}_4 = 0,$$

or

$$p_1 + p_2 + p_3 + p_4 = 0.$$

The first canonical transform(Section 4) is defined as followings:

$$\begin{aligned}
X_1 &= q_2 - q_1, \\
X_2 &= q_4 - q_3, \\
X_3 &= \frac{1}{2}(q_2 - q_1) + \frac{1}{2}(q_4 - q_3) + q_3 - q_2; \\
Y_1 &= -p_1 + \frac{1}{2}(p_1 + p_2), \\
Y_2 &= p_4 + \frac{1}{2}(p_1 + p_2), \\
Y_3 &= -p_1 - p_2;
\end{aligned}$$

$$s = \int_{\tau}^{t}(\frac{m_1m_2}{X_1} + \frac{m_3m_4}{X_2})dt, \qquad (\tau \leq t < t_1)$$

Followed by a second canonical transform:

$$\xi_1 = -X_1Y_1^2, \quad \xi_2 = -X_2Y_2^2, \quad \xi_3 = X_3, \quad \eta_1 = \frac{1}{Y_1}, \quad \eta_2 = \frac{1}{Y_2}, \quad \eta_3 = Y_3;$$

First, we work on the decoupled case: $\xi_3 = \infty, \eta_3 = h = 0$. In proving the existence of analytic solutions of the new variables for this special case, a constant $C = \lim_{s\to 0} \frac{\xi_1-\xi_2}{\xi_1\eta_1^2+\xi_2\eta_2^2}$ appears from our investigation. This constant plays an important role in the existence of analytic solutions and also it gives us a one-parameter family of solutions as s close to the singular time. For simplicity, assume the singular time to be 0. The solutions of the decoupled case have several nice properties:
1. The solutions $\xi_1(s;C) = \xi_2(s;-C)$ and $\eta_1(s;C) = \eta_2(s;-C)$;
2. η_1, η_2 are odd functions and ξ_1 ,ξ_2 are even functions with respect to s;
3. The solutions for $s < 0$ correspond to the behavior of the four bodies before the SBC and the solutions for $s > 0$ correspond to the behavior of the four bodies after the SBC;
4. Although we can choose different values for the constant C for the solution with negative s and for the solution with positive s, the solution will be analytic only when we choose the same constant for both negative s and positive s. In other word, the analytic solution is unique for fixed common constant C on both negative and positive sides of s;

In the coupled case, we use method of majorants to regularize the SBC. But when we come to the solution, several good properties can be observed:
1. The decoupled case is exactly the special case when $\xi_3 = \infty$, $\eta_3 = 0$ and $h = 0$ for the coupled case by matching the constant in two different cases ;
2. In the solution of the coupled case, two constants are founded: one is $\widetilde{d} = -\frac{1}{4}C$, where C is the constant in the decoupled case; the other is ω which is given by the initial conditions of ξ_3, η_3 and h, and shows the effect of the coupling terms to the solutions. And also the coupling term $\widetilde{d}\omega$ will start appearing from the term s^6 in the power series form of X_1 and X_2 ;
3. The motion of the coupled case is very similar to the decoupled case: the solutions can be separated into the decoupled solutions and some power series of ω only up to the power s^4, but the exact solution is NOT a linear combination of the decoupled case with some other movement ;
4. Up to the power s^7, the solution ξ_1, ξ_2, η_1, and η_2 are still symmetric with respect to the new constant $\widetilde{d}$: $\xi_1(s;\widetilde{d}) = \xi_2(s;-\widetilde{d})$ and $\eta_1(s;\widetilde{d}) = \eta_2(s;-\widetilde{d})$;
5. Since ω is fixed for given initial values, but $\widetilde{d}$ will make the solution to be a one-parameter family which is similar to the decoupled case, the analytic solution can only happen if we choose the same common constant $\widetilde{d}$ on both negative and positive sides of s.

2 Preliminaries

(a) Simplified Hamiltonian form

We will use the center of mass integrals to eliminate one pair of variables p_k, q_k from these 8 differential equations, and we will achieve this by taking all the p_k, q_k into new variables x_k, y_k via a suitable canonical transformation. We set

$$p_k = W_{q_k}, \quad x_k = W_{y_k} \qquad (k = 1, 2, 3, 4), \tag{1}$$

where $W(q, y)$ is a generating function whose Jacobian determinant $\mid W_{y_k q_l} \mid$ is not 0. We wish to set up the canonical transformation so that x_1 becomes the distance between P_1 and

P_2, x_2 becomes the distance between P_3 and P_4 and x_3 becomes the distance between P_2 and P_3, while x_4 remains to be the coordinate of P_4, i.e.

$$x_1 = q_2 - q_1, \quad x_2 = q_4 - q_3, \quad x_3 = q_3 - q_2, \quad x_4 = q_4 \qquad (2)$$

and set the generating function as

$$W(q,y) = (q_2 - q_1)y_1 + (q_4 - q_3)y_2 + (q_3 - q_2)y_3 + q_4 y_4.$$

Then $| W_{y_k q_l} |= 1$, and (1) gives a canonical transformation. The first equation of (1) gives

$$p_1 = -y_1, \quad p_2 = y_1 - y_3, \quad p_3 = y_3 - y_2, \quad p_4 = y_2 + y_4.$$

Therefore,

$$y_1 = -p_1, \quad y_2 = -p_1 - p_2 - p_3, \quad y_3 = -p_1 - p_2, \quad y_4 = p_1 + p_2 + p_3 + p_4. \qquad (3)$$

Note $y_4 = 0$, $y_2 = p_4$ since $p_1 + p_2 + p_3 + p_4 = 0$.

And it is easy to see that x_1, x_2, x_3, x_4 are nonnegative but y_1 and y_2 are negative.

In (2), (3) we have the desired transformation, which we see is linear. Since, in addition, it does not depend on t, the new Hamiltonian system is

$$\dot{x_k} = E_{y_k}, \qquad \dot{y_k} = -E_{x_k} \qquad (k = 1,2,3,4), \qquad (4)$$

where $E = T - U$ is regarded as a function of the x_k and y_k. Then

$$T = \frac{1}{2}\sum_{k=1}^{4}\frac{p_k^2}{m_k} = \frac{1}{2}[\frac{y_1^2}{m_1} + \frac{(y_1 - y_3)^2}{m_2} + \frac{(y_3 - y_2)^2}{m_3} + \frac{(y_2 + y_4)^2}{m_4}],$$

$$\begin{aligned} U &= \frac{m_1 m_2}{q_2 - q_1} + \frac{m_1 m_3}{q_3 - q_1} + \frac{m_1 m_4}{q_4 - q_1} + \frac{m_2 m_3}{q_3 - q_2} + \frac{m_2 m_4}{q_4 - q_2} + \frac{m_3 m_4}{q_4 - q_3} \\ &= \frac{m_1 m_2}{x_1} + \frac{m_1 m_3}{x_1 + x_3} + \frac{m_1 m_4}{x_1 + x_2 + x_3} + \frac{m_2 m_3}{x_3} + \frac{m_2 m_4}{x_2 + x_3} + \frac{m_3 m_4}{x_2}. \end{aligned}$$

We know $y_4 = 0$ and

$$\begin{aligned} 0 &= m_1 q_1 + m_2 q_2 + m_3 q_3 + m_4 q_4 \\ &= m_1(x_4 - x_3 - x_2 - x_1) + m_2(x_4 - x_3 - x_2) + m_3(x_4 - x_2) + m_4 x_4, \end{aligned}$$

so that

$$x_4 = \frac{x_1 m_1 + x_3(m_1 + m_2) + x_2(m_1 + m_2 + m_3)}{m_1 + m_2 + m_3 + m_4}.$$

Therefore, we only have to consider the system

$$\dot{x_k} = E_{y_k}, \qquad \dot{y_k} = -E_{x_k} \qquad (k = 1,2,3), \qquad (5)$$

with

$$E = T - U,$$

$$T=\frac{1}{2}[\frac{y_1^2}{m_1}+\frac{(y_1-y_3)^2}{m_2}+\frac{(y_3-y_2)^2}{m_3}+\frac{y_2^2}{m_4}],$$
$$U=\frac{m_1m_2}{x_1}+\frac{m_1m_3}{x_1+x_3}+\frac{m_1m_4}{x_1+x_2+x_3}+\frac{m_2m_3}{x_3}+\frac{m_2m_4}{x_2+x_3}+\frac{m_3m_4}{x_2}.$$

(b) Binary Collision

Assume $t=t_1$ is the collision time.
Then, $x_1\to 0$ and $x_2\to 0$ simultaneously when $t\to t_1$.
The following result is from the work of Belbruno([8]).

Lemma 2.1

$$\lim_{t\to t_1}\frac{x_1}{x_2}=\alpha, \qquad \textit{where}\ \ \alpha=(\frac{m_1+m_2}{m_3+m_4})^{\frac{1}{3}}, \tag{6}$$

and

$$\lim_{t\to t_1}(q_2-q_1)(\dot{q}_2-\dot{q}_1)^2=2(m_1+m_2), \tag{7}$$
$$\lim_{t\to t_1}(q_4-q_3)(\dot{q}_4-\dot{q}_3)^2=2(m_3+m_4). \tag{8}$$

Lemma 2.2 *$x_1y_1^2$ and $x_2y_2^2$ both are finite when $t\to t_1$. Furthermore, $\lim_{t\to t_1}x_1y_1^2$ and $\lim_{t\to t_1}x_2y_2^2$ exist, and also*

$$\lim_{t\to t_1}x_1y_1^2=\lim_{t\to t_1}x_1p_1^2=\frac{2(m_1m_2)^2}{m_1+m_2},$$
$$\lim_{t\to t_1}x_2y_2^2=\lim_{t\to t_1}x_2p_4^2=\frac{2(m_3m_4)^2}{m_3+m_4}.$$

Proof First, we will show both $x_1y_1^2$ and $x_2y_2^2$ are finite.

$$x_1U=m_1m_2+x_1\frac{m_1m_3}{x_1+x_3}+x_1\frac{m_1m_4}{x_1+x_2+x_3}+x_1\frac{m_2m_3}{x_3}+x_1\frac{m_2m_4}{x_2+x_3}+x_1\frac{m_3m_4}{x_2},$$

As $t\to t_1$, we have $x_1\to 0$, $x_2\to 0$,
and x_1+x_3, $x_1+x_2+x_3$, x_3, x_2+x_3, are all positive finite.

$$\therefore \lim_{t\to t_1}x_1U=\lim_{t\to t_1}[m_1m_2+x_1\frac{m_3m_4}{x_2}]=m_1m_2+\alpha m_3m_4.$$

Note that on the phase space of Hamiltonian system (4), $T-U=h$, where h is the Hamiltonian constant.
Therefore, when $t\to t_1$,

$$x_1T=x_1(U+h)\to m_1m_2+\alpha m_3m_4,$$

that is,

$$\frac{1}{2}x_1[\frac{y_1^2}{m_1}+\frac{(y_1-y_3)^2}{m_2}+\frac{(y_3-y_2)^2}{m_3}+\frac{(y_2+y_4)^2}{m_4}]\to m_1m_2+\alpha m_3m_4. \tag{9}$$

In particular, $x_1y_1^2$ and $x_1y_2^2$ are finite at collision. Then, by (6), $x_1y_1^2$ and $x_2y_2^2$ are finite at collision.

Then, we will use the boundedness of $x_1y_1^2$ and $x_2y_2^2$ and Lemma 2.1 to show the existence of the limits of them.
Note that from (7),(8),(9):

$$\lim_{t\to t_1} x_1(\frac{p_1}{m_1}-\frac{p_2}{m_2})^2 = 2(m_1+m_2) \qquad (10)$$

$$\lim_{t\to t_1} x_2(\frac{p_3}{m_3}-\frac{p_4}{m_4})^2 = 2(m_3+m_4) \qquad (11)$$

$$\lim_{t\to t_1} x_1[\frac{y_1^2}{m_1}+\frac{(y_1-y_3)^2}{m_2}+\frac{(y_3-y_2)^2}{m_3}+\frac{(y_2+y_4)^2}{m_4}] = 2(m_1m_2+\alpha m_3m_4) \qquad (12)$$

and

$$p_1+p_2+p_3+p_4=0. \qquad (13)$$

By the boundedness, we can see $x_1p_1^2,\ x_1p_2^2,\ \ x_1p_3^2,\ \ x_1p_4^2$ are all finite when $t\to t_1$.

Because, $p_1+p_2=-y_3$, and by the Hamiltonian system (4),

$$y_3' = -E_{x_3} = \frac{m_1m_3}{(x_1+x_3)^2}+\frac{m_1m_4}{(x_1+x_2+x_3)^2}+\frac{m_2m_3}{(x_3)^2}+\frac{m_2m_4}{(x_2+x_3)^2}$$

For $\tau<t<t_1$, since x_3 is strictly positive, there exists a positive constant B, such that $x_3>B>0$. Therefore,integrate the above identity,

$$y_3(t_1)-y_3(\tau) < \frac{1}{B^2}(t_1-\tau)\cdot(m_1m_3+m_1m_4+m_2m_3+m_2m_4).$$

Since the right hand side of the inequality is finite, $y_3(t_1)$ is bounded above. So p_1+p_2 is finite as t approach t_1.
$\therefore\ x_1p_1^2$, p_1+p_2 are O(1) as t approach t_1.
And then

$$\lim_{t\to t_1} x_1(p_1+p_2)^2 = 0, \qquad \lim_{t\to t_1} x_1p_1(p_1+p_2)=0.$$

Consider (10):

$$2(m_1+m_2) = \lim_{t\to t_1} x_1(\frac{p_1}{m_1}-\frac{p_2}{m_2})^2 = \lim_{t\to t_1} x_1(\frac{p_1}{m_1}+\frac{p_1}{m_2}-\frac{p_1}{m_2}-\frac{p_2}{m_2})^2$$

$$=\lim_{t\to t_1} x_1p_1^2(\frac{1}{m_1}+\frac{1}{m_2})^2+\frac{1}{m_2^2}\lim_{t\to t_1} x_1(p_1+p_2)^2-\frac{2}{m_2}(\frac{1}{m_1}+\frac{1}{m_2})\lim_{t\to t_1} x_1p_1(p_1+p_2)$$

$$=\lim_{t\to t_1} x_1p_1^2(\frac{1}{m_1}+\frac{1}{m_2})^2$$

Therefore, $\lim_{t\to t_1} x_1p_1^2$ exists and the value is $\frac{2(m_1m_2)^2}{m_1+m_2}$.
Similarly, by considering (11), we can get the existence of $\lim_{t\to t_1} x_2p_4^2$ and also

$$\lim_{t\to t_1} x_2y_2^2 = \lim_{t\to t_1} x_2p_4^2 = \frac{2(m_3m_4)^2}{m_3+m_4}. \qquad \blacksquare$$

3 Main Results

In this paper, we showed the regularization of both decoupled case and coupled case. Also we explain the connection of the solutions between those two cases.
In the decoupled case, the following forms give all the possible solutions, which is an one parameter set of solutions.

$$\xi_1^0 = -1 + \widetilde{d}s^2 - \frac{1}{5}\widetilde{d}^2 s^4 - \frac{1}{25}\widetilde{d}^3 s^6 + \frac{7}{1125}\widetilde{d}^4 s^8 + ...$$

$$\xi_2^0 = -1 - \widetilde{d}s^2 - \frac{1}{5}\widetilde{d}^2 s^4 + \frac{1}{25}\widetilde{d}^3 s^6 + \frac{7}{1125}\widetilde{d}^4 s^8 + ...$$

$$\eta_1^0 = \frac{1}{2}s + \frac{\widetilde{d}}{5}s^3 + \frac{\widetilde{d}^2}{10}s^5 + \frac{58\widetilde{d}^3}{1125}s^7 + ...$$

$$\eta_2^0 = \frac{1}{2}s - \frac{\widetilde{d}}{5}s^3 + \frac{\widetilde{d}^2}{10}s^5 - \frac{58\widetilde{d}^3}{1125}s^7 + ...$$

where $\widetilde{d}$ is an arbitrary constant and

$$\widetilde{d} = -\frac{C}{4} = -\frac{1}{4}\lim_{s\to 0}\frac{\xi_1 - \xi_2}{\xi_1\eta_1^2 + \xi_2\eta_2^2}.$$

In the coupled case, all the solutions are:

$$\xi_1 = -1 + (\widetilde{d} - \frac{\omega}{2})s^2 + (-\frac{1}{5}\widetilde{d}^2 + \frac{4}{15}\widetilde{d}\omega - \frac{1}{12}\omega^2)s^4 + \\ (-\frac{61}{2880}\omega^3 + \frac{1}{63}\widetilde{d}\omega^2 + \frac{11}{1050}\widetilde{d}^2\omega - \frac{1}{25}\widetilde{d}^3)s^6 + O(s^8)$$

$$\xi_2 = -1 + (-\widetilde{d} - \frac{\omega}{2})s^2 + (-\frac{1}{5}\widetilde{d}^2 - \frac{4}{15}\widetilde{d}\omega - \frac{1}{12}\omega^2)s^4 + \\ (-\frac{61}{2880}\omega^3 - \frac{1}{63}\widetilde{d}\omega^2 + \frac{11}{1050}\widetilde{d}^2\omega + \frac{1}{25}\widetilde{d}^3)s^6 + O(s^8)$$

$$\eta_1 = \frac{s}{2} + (\frac{1}{5}\widetilde{d} - \frac{\omega}{12})s^3 + (\frac{1}{10}\widetilde{d}^2 - \frac{3}{35}\widetilde{d}\omega + \frac{1}{60}\omega^2)s^5 + \\ (-\frac{6775}{1008000}\omega^3 + \frac{19}{700}\widetilde{d}\omega^2 - \frac{139}{2100}\widetilde{d}^2\omega + \frac{58}{1125}\widetilde{d}^3)s^7 + O(s^9)$$

$$\eta_2 = \frac{s}{2} + (-\frac{1}{5}\widetilde{d} - \frac{\omega}{12})s^3 + (\frac{1}{10}\widetilde{d}^2 + \frac{3}{35}\widetilde{d}\omega + \frac{1}{60}\omega^2)s^5 + \\ (-\frac{6775}{1008000}\omega^3 - \frac{19}{700}\widetilde{d}\omega^2 - \frac{139}{2100}\widetilde{d}^2\omega - \frac{58}{1125}\widetilde{d}^3)s^7 + O(s^9)$$

The above results tell us that
1. In each of the decoupled case and the coupled case, it has one parameter $\widetilde{d}$, which is an arbitrary constant. From the comparison, we can see those two constants are the same. Recall the meaning of C in section 3, $\widetilde{d} = -\frac{C}{4} = -\frac{1}{4}\lim_{s\to 0}\frac{\xi_1 - \xi_2}{\xi_1\eta_1^2 + \xi_2\eta_2^2}$. There is only one way

to make the solution analytic, which is to choose the same $\widetilde{d}$ on the side of $s<0$ and $s>0$. But if we choose $\widetilde{d}_1$ on the collision side $s<0$ and choose a different constant $\widetilde{d}_2$ on the ejection side $s>0$, it is still a solution of the Hamiltonian system and it is not analytic anymore. Consider $x_i=-\xi_1\eta_i^2$, we can see the constant $\widetilde{d}$ will show up from the power s^4 or $t^{\frac{4}{3}}$.

2.The motion of the decoupled case and the coupled case are very similar. Up to the power s^4, the coupled case can be considered as a decoupled case adding another motion which is related to the initial conditions: h, $\widehat{\xi}_3$ and $\widehat{\eta}_3$; on the other hand, because of the mixed term $d\omega$, the coupled solution can NOT be considered exactly as the sum of a decoupled solution and a special solution which has nothing to do with d;

3. Up to the power s^7, the solution ξ_1, ξ_2, η_1, and η_2 are still symmetric with respect to the new constant $\widetilde{d}$: $\xi_1(\widetilde{d})=\xi_2(-\widetilde{d})$ and $\eta_1(\widetilde{d})=\eta_2(-\widetilde{d})$;

4. In the solution of the coupled case, basically there are two constants: $\widetilde{d}=-\frac{1}{4}C$, where C is the constant in the decoupled case; another one ω is given by the initial conditions and it shows the effect of the coupling terms to the solutions, and also the coupling term $\widetilde{d\omega}$ will start appearing from the term s^6 in the power series form of X_1 and X_2;

5. Since ω is fixed for given initial values, but $\widetilde{d}$ will make the solution to be a one-parameter family which is similar to the decoupled case, and the analytic solution can ONLY happen if we choose the same common constant $\widetilde{d}$ on both negative and positive sides of s.

4 Decoupled case with all masses equal to 1

Replace time variable t by the new independent variable

$$s=\int_\tau^t\left(\frac{m_1m_2}{x_1}+\frac{m_3m_4}{x_2}\right)dt,\qquad (\tau\leq t<t_1)$$

Siegel and Moser([6]) have shown that $\int_\tau^{t_1}U dt$ is finite, then $s_1=\int_\tau^{t_1}(\frac{m_1m_2}{x_1}+\frac{m_3m_4}{x_2})dt$ is also finite.

Denote $\frac{dx_k}{ds}$ by x_k' and $\frac{dy_k}{ds}$ by y_k'.Then the system (5) becomes

$$x_k'=\frac{1}{\frac{m_1m_2}{x_1}+\frac{m_3m_4}{x_2}}E_{y_k},\qquad y_k'=-\frac{1}{\frac{m_1m_2}{x_1}+\frac{m_3m_4}{x_2}}E_{x_k}\qquad k=1,2,3 \tag{18}$$

Set $F=\frac{1}{\frac{m_1m_2}{x_1}+\frac{m_3m_4}{x_2}}(E-h)=\frac{1}{\frac{m_1m_2}{x_1}+\frac{m_3m_4}{x_2}}(T-U-h)$, where $E=T-U=h$. Then for the solution of system (5) on the energy surface $E=h$, we have

$$F_{x_k}=\frac{1}{\frac{m_1m_2}{x_1}+\frac{m_3m_4}{x_2}}E_{x_k},\qquad F_{y_k}=\frac{1}{\frac{m_1m_2}{x_1}+\frac{m_3m_4}{x_2}}E_{y_k}.$$

Consequently, for the solution of system (5) on the energy surface $E=h$, (18) can be written as

$$x_k'=F_{y_k},\qquad y_k'=-F_{x_k}\qquad k=1,2,3$$

with $F = \frac{1}{\frac{m_1 m_2}{x_1}+\frac{m_3 m_4}{x_2}}(T-U-h)$.
If x_k and y_k are solutions of (5) on the energy surface $E=h$, F is a constant with respect to s because

$$F' = \sum_{k=1}^{3}(F_{x_k}x_k' + F_{y_k}y_k') = 0.$$

For simplification, assume $y_3 = 0$, $x_3 = \infty$, $h = 0$, and $m_1 = m_2 = m_3 = m_4 = 1$.
we choose

$$F = \frac{y_1^2+y_2^2}{\frac{1}{x_1}+\frac{1}{x_2}}. \tag{19}$$

To solve the Hamiltonian system,

$$x_k' = F_{y_k}, \qquad y_k' = -F_{x_k} \qquad k = 1,2 \tag{20}$$

with $F = \frac{y_1^2+y_2^2}{\frac{1}{x_1}+\frac{1}{x_2}}$, we are going to construct a canonical transform.
(a)**Relationship between x_k and y_k**

First, let's write (20) into explicit forms:

$$x_1' = \frac{2y_1}{\frac{1}{x_1}+\frac{1}{x_2}} = \frac{2y_1x_1x_2}{x_1+x_2}, \tag{21}$$

$$y_1' = -\frac{y_1^2+y_2^2}{(\frac{1}{x_1}+\frac{1}{x_2})^2}\frac{1}{x_1^2} = -\frac{x_2}{x_1(x_1+x_2)}F, \tag{22}$$

$$x_2' = \frac{2y_2}{\frac{1}{x_1}+\frac{1}{x_2}} = \frac{2y_2x_1x_2}{x_1+x_2}, \tag{23}$$

$$y_2' = -\frac{y_1^2+y_2^2}{(\frac{1}{x_1}+\frac{1}{x_2})^2}\frac{1}{x_2^2} = -\frac{x_1}{x_2(x_1+x_2)}F. \tag{24}$$

Think y_k as a function of x_k(k=1,2).

Lemma 4.1 *If $\{x_1, x_2, y_1, y_2\}$ is the solution for the above system (21)-(24), there exists a constant C such that*

$$y_1^2 = \frac{F}{x_1} + C, \qquad \textit{and} \qquad y_2^2 = \frac{F}{x_2} - C,$$

$$\textit{or} \qquad C = \frac{x_1y_1^2 - x_2y_2^2}{x_1+x_2}.$$

Proof By (21) and (22):

$$\frac{dy_1}{dx_1} = \frac{y_1'}{x_1'} = -\frac{F}{2y_1x_1^2}$$

Then separate the variables:

$$\int 2y_1 dy_1 = \int \frac{-F}{x_1^2} dx_1$$

Therefore,

$$y_1^2 = \frac{F}{x_1} + C, \quad or \quad x_1 y_1^2 - Cx_1 = F \tag{25}$$

where C is a constant, which depends on the initial conditions.
By a similar process, we have

$$y_2^2 = \frac{F}{x_2} + C_1, \quad or \quad x_2 y_2^2 - C_1 x_2 = F \tag{26}$$

where C_1 is another constant, which depends on the initial conditions, too.
Add (25) and (26) together, by using (19):

$$y_1^2 + y_2^2 = \frac{F}{x_1} + \frac{F}{x_2} + C + C_1$$

$$= y_1^2 + y_2^2 + C + C_1$$

Therefore, $C_1 = -C$. Then we can rewrite (26) as

$$y_2^2 = \frac{F}{x_2} - C, \quad or \quad x_2 y_2^2 + Cx_2 = F. \tag{27}$$

(25)-(27):

$$x_1 y_1^2 - x_2 y_2^2 - C(x_1 + x_2) = 0,$$

$$C = \frac{x_1 y_1^2 - x_2 y_2^2}{x_1 + x_2}. \qquad \blacksquare$$

(b)**Canonical Transformation**

As we know, at the collision time t_1, $y_1 \to \infty$ and $y_2 \to \infty$. It would be nice if we can make a canonical transformation without singularity at $t = t_1$.
From the two-body problem, we have a transformation as $\eta_k = \frac{1}{y_k}$. Similarly, we want to use this part to generate our canonical transformation.

$$y_1 = \frac{1}{\eta_1}, \qquad y_2 = \frac{1}{\eta_2}.$$

Let $y = (y_1, y_2)^T$, $b(\eta) = (\frac{1}{\eta_1}, \frac{1}{\eta_2})^T$ and $x = (x_1, x_2)^T$. Assume the generating function $V = V(x, \eta)$. The canonical transformation is given by

$$y = V_x(x, \eta), \qquad \xi = V_\eta(x, \eta).$$

Hence,

$$y = V_x(x, \eta) = b(\eta)$$

Therefore,

$$V(x,\eta) = < b(\eta), x > + g(\eta)$$

Then

$$\xi = V_\eta(x,\eta) = b_\eta^T(\eta)\cdot x + g_\eta(\eta)$$

So

$$x = [b_\eta^T(\eta)]^{-1}\cdot(\xi - g_\eta(\eta)).$$

In particular, let $g(\eta) = 0$.
Since

$$b_\eta(\eta) = \begin{pmatrix} -\frac{1}{\eta_1^2} & 0 \\ 0 & -\frac{1}{\eta_2^2} \end{pmatrix} = b_\eta^T(\eta),$$

we can write down the canonical transformation as

$$\xi_1 = -x_1 y_1^2, \qquad \xi_2 = -x_2 y_2^2, \qquad \eta_1 = \frac{1}{y_1}, \qquad \eta_2 = \frac{1}{y_2};$$

$$x_1 = -\xi_1\eta_1^2, \qquad x_2 = -\xi_2\eta_2^2, \qquad y_1 = \frac{1}{\eta_1}, \qquad y_2 = \frac{1}{\eta_2}.$$

And the new hamiltonian system is going to be

$$\xi_k' = F_{\eta_k}, \qquad \eta_k' = -F_{\xi_k} \qquad k = 1,2 \tag{28}$$

with

$$F = -\frac{\xi_1\xi_2(\eta_1^2+\eta_2^2)}{\xi_1\eta_1^2+\xi_2\eta_2^2} = -\frac{\eta_1^2+\eta_2^2}{\frac{\eta_1^2}{\xi_2}+\frac{\eta_2^2}{\xi_1}}. \tag{29}$$

(c)**Meaning of C**

Since we only know the behavior at time $t = t_1$ or $s = s_1$.We may think $s = s_1$ as the initial time for the hamiltonian system (28).
Without loss of generality, let $s_1 = 0$. We will consider the following two differential equations

$$\eta_1 = -F_{\xi_1}, \qquad \eta_2 = -F_{\xi_2}$$

with initial conditions:

$$\eta_1(0) = \eta_2(0) = 0, \qquad \xi_1(0) = \xi_2(0) = -F.$$

(25) and (27) can be rewritten in terms of ξ_k and η_k:

$$-\xi_1 + C\xi_1\eta_1^2 = -\xi_2 - C\xi_2\eta_2^2 = F,$$

then

$$\xi_1 = \frac{F}{-1+C\eta_1^2}, \qquad \xi_2 = \frac{F}{-1-C\eta_2^2}.$$

Differentiate (29),

$$F_{\xi_1} = \frac{-\xi_2(\eta_1^2+\eta_2^2)}{\xi_1\eta_1^2+\xi_2\eta_2^2} + \frac{\xi_1\xi_2(\eta_1^2+\eta_2^2)\eta_1^2}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2}$$

$$= \frac{F}{\xi_1} + \frac{F^2}{\xi_1 \xi_2} \cdot \frac{\eta_1^2}{\eta_1^2 + \eta_2^2} = -\frac{F^2}{\xi_1^2} \cdot \frac{\eta_2^2}{\eta_1^2 + \eta_2^2}$$

$$= -(-1 + C\eta_1^2)^2 \cdot \frac{\eta_2^2}{\eta_1^2 + \eta_2^2},$$

and similarly,

$$F_{\xi_2} = \frac{-\xi_1(\eta_1^2 + \eta_2^2)}{\xi_1\eta_1^2 + \xi_2\eta_2^2} + \frac{\xi_1\xi_2(\eta_1^2 + \eta_2^2)\eta_2^2}{(\xi_1\eta_1^2 + \xi_2\eta_2^2)^2}$$

$$= \frac{F}{\xi_2} + \frac{F^2}{\xi_1 \xi_2} \cdot \frac{\eta_2^2}{\eta_1^2 + \eta_2^2} = -\frac{F^2}{\xi_2^2} \cdot \frac{\eta_1^2}{\eta_1^2 + \eta_2^2}$$

$$= -(-1 - C\eta_2^2)^2 \cdot \frac{\eta_1^2}{\eta_1^2 + \eta_2^2}.$$

Therefore,

$$\eta_1' = (-1 + C\eta_1^2)^2 \cdot \frac{\eta_2^2}{\eta_1^2 + \eta_2^2} \qquad (30)$$

$$\eta_2' = (-1 - C\eta_2^2)^2 \cdot \frac{\eta_1^2}{\eta_1^2 + \eta_2^2} \qquad (31)$$

Actually, C is an arbitrary constant.
When $C = 0$, it is the most special case. The solution for the equations (30) and (31) with initial conditions $\eta_1(0) = \eta_2(0) = 0$ is

$$\eta_1 = \eta_2 = \frac{s}{2}.$$

Then

$$\xi_1 = \frac{F}{-1 + C\eta_1^2} = -F, \qquad \xi_2 = \frac{F}{-1 - C\eta_2^2} = -F.$$

Therefore, $x_1 = x_2$ and $y_1 = y_2$.
So when $C = 0$, the motions of this two decoupled system are exactly the same.

When $C > 0$, that is $C = \frac{x_1y_1^2 - x_2y_2^2}{x_1+x_2} > 0$, hence $x_1y_1^2 > x_2y_2^2$.
Consider the initial conditions. Assume x_1, x_2, y_1 and y_2 are the initial values for the decoupled system. If $x_1 < x_2$, which means the distance between the two object P_1 and P_2 is less than the distance between the two object P_3 and P_4, then $y_1 > y_2$ because $x_1y_1^2 > x_2y_2^2$. That is, the initial velocity of P_1 is also greater than the initial velocity of P_2. Note that for each collision system, the force between the objects only depends on the relative distance, and then by Newton's second law, the acceleration for P_1 or P_2 is greater than the acceleration for P_3 or P_4. So it is impossible for these two collisions to happen at the same time. Contradiction!
therefore, when $C > 0$, we can get

$$x_1 > x_2 \quad \text{and} \quad y_1 > y_2.$$

By a similar argument, we know when $C < 0$, we have

$$x_1 < x_2 \quad \text{and} \quad y_1 < y_2.$$

(d)**Apply C to the decoupled system**

First, we want to show that the constant C can also be derived directly from the following initial value problem:

$$\xi_1' = -\frac{2\eta_1\eta_2^2(\frac{1}{\xi_1}-\frac{1}{\xi_2})}{(\frac{\eta_1^2}{\xi_2}+\frac{\eta_2^2}{\xi_1})^2} = \frac{2\xi_1\xi_2\eta_1\eta_2^2(\xi_1-\xi_2)}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2},$$

$$\xi_2' = -\frac{2\eta_2\eta_1^2(\frac{1}{\xi_2}-\frac{1}{\xi_1})}{(\frac{\eta_1^2}{\xi_2}+\frac{\eta_2^2}{\xi_1})^2} = -\frac{2\xi_1\xi_2\eta_2\eta_1^2(\xi_1-\xi_2)}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2},$$

$$\eta_1' = \frac{F^2}{\xi_1^2}\cdot\frac{\eta_2^2}{\eta_1^2+\eta_2^2},$$

$$\eta_2' = \frac{F^2}{\xi_2^2}\cdot\frac{\eta_1^2}{\eta_1^2+\eta_2^2},$$

$$\xi_1(0)=\xi_2(0)=-F, \qquad \eta_1(0)=\eta_2(0)=0.$$

Let $f(s) = \frac{\frac{F}{\xi_1}-\frac{F}{\xi_2}}{\eta_1^2+\eta_1^2}$, because $F = -\frac{\eta_1^2+\eta_2^2}{\frac{\eta_1^2}{\xi_2}+\frac{\eta_2^2}{\xi_1}}$,

$$f(s) = -\frac{\eta_1^2+\eta_2^2}{\frac{\eta_1^2}{\xi_2}+\frac{\eta_2^2}{\xi_1}}\cdot\frac{\frac{1}{\xi_1}-\frac{1}{\xi_2}}{\eta_1^2+\eta_1^2} = \frac{\xi_1-\xi_2}{\xi_1\eta_1^2+\xi_2\eta_2^2}.$$

Lemma 4.2 *If $\{\xi_1,\xi_2,\eta_1,\eta_2\}$ is the solution for the above system with the initial conditions, then $f(s)$ is a constant with respect to s.*

Proof

$$\frac{df}{ds} = \frac{\partial f}{\partial\xi_1}\cdot\xi_1' + \frac{\partial f}{\partial\xi_2}\cdot\xi_2' + \frac{\partial f}{\partial\eta_1}\cdot\eta_1' + \frac{\partial f}{\partial\eta_2}\cdot\eta_2'$$

$$= \frac{\xi_2(\eta_1^2+\eta_2^2)}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2}\cdot\frac{2\xi_1\xi_2\eta_1\eta_2^2(\xi_1-\xi_2)}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2} + \frac{-\xi_1(\eta_1^2+\eta_2^2)}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2}\cdot\frac{-2\xi_1\xi_2\eta_2\eta_1^2(\xi_1-\xi_2)}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2}$$

$$+\frac{-2\xi_1\eta_1(\xi_1-\xi_2)}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2}\cdot\frac{\xi_2^2\eta_2^2(\eta_1^2+\eta_2^2)}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2} + \frac{-2\xi_2\eta_2(\xi_1-\xi_2)}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2}\cdot\frac{\xi_1^2\eta_1^2(\eta_1^2+\eta_2^2)}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2}$$

$$= 0.$$

Therefore, $f(s)$ is a constant. ∎

Lemma 4.3 *Let $f(s) = C_2$, then $C_2 = C$, where C is the constant in Lemma 4.1.*

Proof From the initial value problem,

$$\frac{d\xi_1}{d\eta_1} = \frac{\xi_1'}{\eta_1'} = \frac{2\xi_1\xi_2\eta_1\eta_2^2(\xi_1-\xi_2)}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2} \cdot \frac{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2}{\xi_2^2\eta_2^2(\eta_1^2+\eta_2^2)} = \frac{2\xi_1\eta_1(\xi_1-\xi_2)}{\xi_2(\eta_1^2+\eta_2^2)}$$

Note that $F = -\frac{\eta_1^2+\eta_2^2}{\frac{\eta_1^2}{\xi_2}+\frac{\eta_2^2}{\xi_1}}$, then $\eta_1^2+\eta_2^2 = -F\cdot(\frac{\eta_1^2}{\xi_2}+\frac{\eta_2^2}{\xi_1})$, so

$$\frac{d\xi_1}{d\eta_1} = \frac{2\xi_1\eta_1(\xi_1-\xi_2)}{\xi_2(\eta_1^2+\eta_2^2)}$$

$$= -\frac{1}{F}\cdot\frac{2\xi_1\eta_1(\xi_1-\xi_2)}{\xi_2(\frac{\eta_1^2}{\xi_2}+\frac{\eta_2^2}{\xi_1})}$$

$$= -\frac{1}{F}\cdot 2\xi_1^2\eta_1\cdot\frac{\xi_1-\xi_2}{\xi_1\eta_1^2+\xi_2\eta_2^2} = -\frac{C_2}{F}\cdot 2\xi_1^2\eta_1$$

Separate the variable,

$$\frac{d\xi_1}{-\xi_1^2} = \frac{C_2}{F}\cdot 2\eta_1 d\eta_1$$

integrate both sides,

$$\frac{1}{\xi_1} = \frac{C_2}{F}\eta_1^2 + C_3,$$

where C_3 is a constant.
By the initial condition,

$$-\frac{1}{F} = C_3.$$

Therefore,

$$\frac{1}{\xi_1} = \frac{C_2}{F}\eta_1^2 - \frac{1}{F}$$

that is

$$F = -\xi_1 + C_2\xi_1\eta_1^2.$$

But from the definition of transformation, we have

$$F = -\xi_1 + C\xi_1\eta_1^2,$$

then

$$C = C_2. \quad \blacksquare$$

Therefore, we can use the constant C in the new hamiltonian system. And note that C acts as a first integral in the new hamiltonian system, which does not depend on the constant F. Rewrite the system:

$$\xi_1' = -\frac{2\eta_1\eta_2^2(\frac{1}{\xi_1}-\frac{1}{\xi_2})}{(\frac{\eta_1^2}{\xi_2}+\frac{\eta_2^2}{\xi_1})^2} = -2CF\frac{\eta_1\eta_2^2}{\eta_1^2+\eta_2^2},$$

$$\xi_2' = -\frac{2\eta_2\eta_1^2(\frac{1}{\xi_2}-\frac{1}{\xi_1})}{(\frac{\eta_1^2}{\xi_2}+\frac{\eta_2^2}{\xi_1})^2} = 2CF\frac{\eta_1^2\eta_2}{\eta_1^2+\eta_2^2},$$
$$\eta_1' = \frac{F^2}{\xi_1^2}\cdot\frac{\eta_2^2}{\eta_1^2+\eta_2^2},$$
$$\eta_2' = \frac{F^2}{\xi_2^2}\cdot\frac{\eta_1^2}{\eta_1^2+\eta_2^2}.$$

By observation, we have

$$\eta_1'\xi_1 + \eta_2'\xi_2 = -F, \qquad (32)$$
$$\eta_1\xi_1' + \eta_2\xi_2' = 0, \qquad (33)$$
$$\eta_1'\xi_1^2 + \eta_2'\xi_2^2 = F^2, \qquad (34)$$
$$\eta_1'\eta_1^2\xi_1^2 - \eta_2'\eta_2^2\xi_2^2 = 0, \qquad (35)$$
$$\frac{\xi_1'}{\eta_1} - \frac{\xi_2'}{\eta_2} = -2CF. \qquad (36)$$

By (32) and (33),

$$(\eta_1\xi_1 + \eta_2\xi_2)' = -F$$

Since when $s=0$, $\eta_1 = \eta_2 = 0$, therefore by integrating both sides,

$$\eta_1\xi_1 + \eta_2\xi_2 = -Fs$$

By the identities $-\xi_1 + C\xi_1\eta_1^2 = -\xi_2 - C\xi_2\eta_2^2 = F$,

$$\frac{\eta_1}{1-C\eta_1^2} + \frac{\eta_2}{1+C\eta_2^2} = s, \qquad (37)$$

and also the differential equations of η_1 and η_2 will be

$$\eta_1' = (-1+C\eta_1^2)^2\cdot\frac{\eta_2^2}{\eta_1^2+\eta_2^2}$$

$$\eta_2' = (-1-C\eta_2^2)^2\cdot\frac{\eta_1^2}{\eta_1^2+\eta_2^2}$$

Because for $C<0$, the equations are

$$\eta_1' = (-1-\mid C\mid\eta_1^2)^2\cdot\frac{\eta_2^2}{\eta_1^2+\eta_2^2}$$
$$\eta_2' = (-1+\mid C\mid\eta_2^2)^2\cdot\frac{\eta_1^2}{\eta_1^2+\eta_2^2}$$

Note that the solutions $\{\eta_1,\eta_2\}$ of the above two equations are the same as the solutions $\{\eta_2,\eta_1\}$ of (30) and (31) with positive C.

Without loss of generality, we can assume that $C>0$.

Lemma 4.4 *Let $\{\eta_1, \eta_2\}$ is the solution for*

$$\eta_1' = (-1 + C\eta_1^2)^2 \cdot \frac{\eta_2^2}{\eta_1^2 + \eta_2^2},$$

$$\eta_2' = (-1 - C\eta_2^2)^2 \cdot \frac{\eta_1^2}{\eta_1^2 + \eta_2^2}.$$

Define $N_1(s) = C^{\frac{1}{2}}\eta_1(\frac{s}{C^{\frac{1}{2}}})$ and $N_2(s) = C^{\frac{1}{2}}\eta_2(\frac{s}{C^{\frac{1}{2}}})$. Then

$$\tanh^{-1}(N_1) + \tan^{-1}(N_2) = s,$$

$$\frac{N_1}{1 - N_1^2} + \frac{N_2}{1 + N_2^2} = s.$$

Proof Consider the fraction between the above two equations:

$$\frac{\eta_1'}{\eta_2'} = \frac{(-1 + C\eta_1^2)^2}{(-1 - C\eta_2^2)^2} \cdot \frac{\eta_2^2}{\eta_1^2}$$

Separating the variables and integrating both sides:

$$\frac{\eta_1^2}{(-1 + C\eta_1^2)^2} d\eta_1 = \frac{\eta_2^2}{(1 + C\eta_2^2)^2} d\eta_2$$

$$-\frac{1}{2C}\frac{\eta_1}{-1 + C\eta_1^2} - \frac{1}{2}\frac{\tanh^{-1}(C^{\frac{1}{2}}\eta_1)}{C^{\frac{3}{2}}} = -\frac{1}{2C}\frac{\eta_2}{1 + C\eta_2^2} + \frac{1}{2}\frac{\tan^{-1}(C^{\frac{1}{2}}\eta_2)}{C^{\frac{3}{2}}} + D$$

where D is a constant.
By the initial condition $\eta_1(0) = \eta_2(0) = 0$,

$$D = 0.$$

Simplify the above identity of η_1 and η_2:

$$\frac{C^{\frac{1}{2}}\eta_1}{-1 + C\eta_1^2} + \tanh^{-1}(C^{\frac{1}{2}}\eta_1) = \frac{C^{\frac{1}{2}}\eta_2}{1 + C\eta_2^2} - \tan^{-1}(C^{\frac{1}{2}}\eta_2) \tag{38}$$

Therefore, combine (38) and (37),

$$\tanh^{-1}(C^{\frac{1}{2}}\eta_1) + \tan^{-1}(C^{\frac{1}{2}}\eta_2) = \frac{C^{\frac{1}{2}}\eta_2}{1 + C\eta_2^2} + \frac{C^{\frac{1}{2}}\eta_1}{1 - C\eta_1^2} = C^{\frac{1}{2}}s. \tag{39}$$

Then it is easy to express η_2 in terms of η_1:

$$\eta_2 = C^{-\frac{1}{2}}\tan[C^{\frac{1}{2}}s - \tanh^{-1}(C^{\frac{1}{2}}\eta_1)].$$

Let$N_1(s) = C^{\frac{1}{2}}\eta_1(\frac{s}{C^{\frac{1}{2}}})$, $N_2(s) = C^{\frac{1}{2}}\eta_2(\frac{s}{C^{\frac{1}{2}}})$.
Then the equations of η_1 and η_2 can be changed to equations of N_1 and N_2:

$$N_1' = (1 - N_1^2)^2 \cdot \frac{N_2^2}{N_1^2 + N_2^2}$$

$$N_2' = (1 + N_2^2)^2 \cdot \frac{N_1^2}{N_1^2 + N_2^2}$$

Therefore, by (39),

$$\tanh^{-1}(N_1) + \tan^{-1}(N_2) = s, \tag{40}$$

and

$$\frac{N_1}{1 - N_1^2} + \frac{N_2}{1 + N_2^2} = s. \tag{41}$$

∎

(e)**Existence, Uniqueness and Analytic Properties**

Consider the equations

$$N_1' = (1 - N_1^2)^2 \cdot \frac{N_2^2}{N_1^2 + N_2^2},$$

$$N_2' = (1 + N_2^2)^2 \cdot \frac{N_1^2}{N_1^2 + N_2^2},$$

with the initial conditions

$$N_1(0) = N_2(0) = 0.$$

Theorem 4.5 *The above system has analytic solutions $(N_1(s), N_2(s))$ as s approaches 0.*

To prove the above theorem, we need to introduce several propositions.

Proposition 4.6 *Assume N_1 and N_2 satisfy*

$$\tanh^{-1}(N_1) + \tan^{-1}(N_2) = \frac{N_1}{1 - N_1^2} + \frac{N_2}{1 + N_2^2},$$

and $(N_1, N_2) = (0, 0)$ is a point on the curve given by the above equation, then N_1 is a real analytic function of N_2 in a small neighborhood of $N_2 = 0$.

Proof:Basically, we will apply the implicit function theorem.
Compare (40) and (41), we get a equation of N_1 and N_2:

$$\tanh^{-1}(N_1) + \tan^{-1}(N_2) = \frac{N_1}{1 - N_1^2} + \frac{N_2}{1 + N_2^2}$$

or

$$-\tanh^{-1}(N_1) - \tan^{-1}(N_2) + \frac{N_1}{1 - N_1^2} + \frac{N_2}{1 + N_2^2} = 0$$

Let$G(N_1,N_2) = -\tanh^{-1}(N_1) - \tan^{-1}(N_2) + \frac{N_1}{1-N_1^2} + \frac{N_2}{1+N_2^2}$, of course there exists a small neighborhood V of (0,0), such that $G(N_1,N_2)$ is analytic in V with respect to (N_1,N_2). The Taylor series of $G(N_1,N_2)$ at $(0,0)$ is

$$G(N_1,N_2) = \sum_{n=1}^{\infty} \frac{2n}{2n+1} N_1^{2n+1} + \sum_{n=1}^{\infty} \frac{2n(-1)^n}{2n+1} N_2^{2n+1}$$

We know N_1 and N_2 satisfy $G(N_1,N_2)=0$. When applying the implicit theorem, we will see that $\frac{\partial G}{\partial N_1}(0,0)=0$. So we need to modify the theorem a little bit.

Proposition 4.7 *(Extended Implicit Function Theorem)*
Denote $\frac{\partial G}{\partial N_1}$ *by* G'_{N_1}*, the second partial derivative* $\frac{\partial^2 G}{\partial N_1^2}$ *by* G''_{N_1}*, and the third partial derivative* $\frac{\partial^3 G}{\partial N_1^3}$ *by* G'''_{N_1}*. If*

$$G(0,0) = 0,$$
$$G'_{N_1}(0,N_2) = 0,$$
$$G''_{N_1}(0,N_2) = 0,$$
$$G'''_{N_1}(0,N_2) \neq 0,$$

then there exist intervals $I = (-\delta_1, \delta_1)$ *and* $J = (-\delta_2,\delta_2)$ *and a unique function* g*, such that*

$$g : J \longrightarrow I, \qquad N_2 \mapsto N_1 = g(N_2).$$

Proof: Differentiate the Taylor series of $G(N_1,N_2)$ with respect to N_1 three times, we can find some good properties about the partial derivatives of $G(N_1,N_2)$:

$$G'_{N_1}(0,N_2) = 0$$

$$G''_{N_1}(0,N_2) = 0$$

$$G'''_{N_1}(0,N_2) = 4 \neq 0$$

Because $G'''_{N_1}(0,N_2) = 4 > 0$ and G'''_{N_1} is continuous, there exist a rectangular area R: $\mid N_1 \mid < \delta_1$, $\mid N_2 \mid < \delta'_2$, such that the closure $\overline{R} \subset V$ and

$$m = \min_{(N_1,N_2)\in R} G'''_{N_1}(N_1,N_2) > 0.$$

Since $G''_{N_1}(0,N_2)=0$ and $G''_{N_1}(N_1,N_2)$ is continuous and strictly increasing with respect to N_1,

$$G''_{N_1}(N_1,N_2) > 0 \quad for \ \ 0 < N_1 < \delta_1$$

and

$$G''_{N_1}(N_1,N_2) < 0 \quad for \ \ -\delta_1 < N_1 < 0.$$

By the above result, $G'_{N_1}(N_1,N_2)$ is strictly increasing with respect to N_1 when $0 < N_1 < \delta_1$ and $G'_{N_1}(N_1,N_2)$ is strictly decreasing with respect to N_1 in $(-\delta_1, 0)$. Note that $G'_{N_1}(0,N_2) = 0$,

$$G'_{N_1}(N_1,N_2) > 0 \quad for \ \ 0 < N_1 < \delta_1$$

$$G'_{N_1}(N_1,N_2)>0 \quad for \quad -\delta_1<N_1<0$$

that is

$$G'_{N_1}(N_1,N_2)>0 \quad for \quad -\delta_1<N_1<\delta_1, \quad N_1\neq 0.$$

Because $G(0,0)=0$, $G'_{N_1}(N_1,N_2)>0$ when $N_1\neq 0$,then

$$G(-\delta_1,0)<0, \qquad G(\delta_1,0)>0.$$

By the continuity of $G(N_1,N_2)$, there exists $0<\delta_2<\delta_2'$, such that when $\mid N_2 \mid<\delta_2$,

$$G(-\delta_1,N_2)<0, \qquad G(\delta_1,N_2)>0.$$

Consider the intervals $I=(-\delta_1,\delta_1)$ and $J=(-\delta_2,\delta_2)$. For any point N_2 in J, the function $G(N_1,N_2)$ is strictly increasing in I, by the intermediate value theorem for continuous function, there exists only one $N_1\in I$ such that $G(N_1,N_2)=0$. That means, for any given $N_2\in J$, according to $G(N_1,N_2)=0$, we can always find only one $N_1\in I$ corresponds to N_2. By the definition of function, there exist a function g such that

$$g: J\longrightarrow I, \qquad N_2\mapsto N_1=g(N_2).$$

Next we need to show g is unique.
If there exist

$$\widetilde{g}: J\longrightarrow I, \qquad N_2\mapsto N_1=\widetilde{g}(N_2)$$

satisfying $G(\widetilde{g}(N_2),N_2)=0$, $(N_2\in J)$, then $g(N_2)=\widetilde{g}(N_2)$ $(N_2\in J)$.
Hence, so far we have proven the existence and uniqueness of N_1 as a function of N_2 which satisfy $G(N_1,N_2)=0$. ■

Proof of Proposition 4.6: By Prop. 3.3, the existence and uniqueness are guarantee. If we can show N_1 is an analytic function of N_2, then we are done.

Consider

$$-\tanh^{-1}(N_1)-\tan^{-1}(N_2)+\frac{N_1}{1-N_1^2}+\frac{N_2}{1+N_2^2}=0$$

Since each function on the left hand side of the equality is analytic close to 0, we can find their Taylor expansions for (N_1,N_2) in a small interval of $(0,0)$:

$$\sum_{n=1}^{\infty}\frac{2n}{2n+1}N_1^{2n+1}+\sum_{n=1}^{\infty}\frac{2n(-1)^n}{2n+1}N_2^{2n+1}=0,$$

that is

$$\sum_{n=1}^{\infty}\frac{2n}{2n+1}N_1^{2n+1}=\sum_{n=1}^{\infty}\frac{2n(-1)^{n+1}}{2n+1}N_2^{2n+1}$$

$${N_1}^3(\frac{2}{3}+\sum_{n=2}^{\infty}\frac{2n}{2n+1}N_1^{2n-2})={N_2}^3(\frac{2}{3}+\sum_{n=2}^{\infty}\frac{2n(-1)^{n+1}}{2n+1}N_2^{2n-2}).$$

For simplicity, let

$$h_1(N_1)=\frac{2}{3}+\sum_{n=2}^{\infty}\frac{2n}{2n+1}N_1^{2n-2}$$

and

$$h_2(N_2) = \frac{2}{3} + \sum_{n=2}^{\infty} \frac{2n(-1)^{n+1}}{2n+1} N_2^{2n-2},$$

by ratio test, we can see that $h_1(N_1)$ and $h_2(N_2)$ both are analytic in a neighborhood of 0 and the radius of convergence is 1.

In calculus, we know that when $r \neq 0$, $(1+x)^r$ is analytic for $x \in (-1,1)$ and the Taylor series at 0 is

$$(1+x)^r = \sum_{k=0}^{\infty} \frac{r[r-1][r-2]...[r-(k-1)]}{k!} x^k.$$

Denote $\frac{3}{2}\sum_{n=2}^{\infty} \frac{2n}{2n+1} N_1^{2n-2}$ by $u_1(N_1)$ and $\frac{3}{2}\sum_{n=2}^{\infty} \frac{2n(-1)^{n+1}}{2n+1} N_1^{2n-2}$ by $u_2(N_2)$,

$$[\frac{3}{2}h_1(N_1)]^{\frac{1}{3}} = [1+u_1]^{\frac{1}{3}}$$

is an analytic function of u_1. Because the composition of two analytic functions is still analytic, then $[\frac{3}{2}h_1(N_1)]^{\frac{1}{3}}$ is analytic for N_1 in a small neighborhood of 0. So does $[h_1(N_1)]^{\frac{1}{3}}$. Similarly, $[h_2(N_2)]^{\frac{1}{3}}$ is analytic for N_2 in a small neighborhood of 0.

Because

$$N_1^3 \cdot h_1(N_1) = N_2^3 \cdot h_2(N_2)$$

do the cubic roots both sides,

$$N_1 \cdot [h_1(N_1)]^{\frac{1}{3}} = N_2 \cdot [h_2(N_2)]^{\frac{1}{3}}.$$

By the above argument, both sides are analytic. Let

$$\Gamma(N_1, N_2) = N_1 \cdot [h_1(N_1)]^{\frac{1}{3}} - N_2 \cdot [h_2(N_2)]^{\frac{1}{3}},$$

then $\Gamma(N_1, N_2)$ is analytic with respect to (N_1, N_2) in a small neighbor hood of $(0,0)$.

In order to apply the analytic implicit function theorem, we need to check the conditions:

$$\Gamma(0,0) = 0,$$

$$\begin{aligned}
\frac{\partial \Gamma}{\partial N_1}(0,0) &= [h_1(N_1)]^{\frac{1}{3}} + N_1 \cdot \frac{1}{3}[h_1(N_1)]^{-\frac{2}{3}} \cdot h_1'(N_1) \mid_{N_1=0} \\
&= (\frac{2}{3})^{\frac{1}{3}} + 0 \cdot \frac{1}{3} \cdot (\frac{2}{3})^{-\frac{2}{3}} \cdot 0 \\
&= (\frac{2}{3})^{\frac{1}{3}} \neq 0,
\end{aligned}$$

by Cauchy's analytic implicit theorem, there exists $r_0 > 0$,and a power series

$$N_1(N_2) = \sum_{i=0}^{\infty} a_i N_2^i$$

such that $N_1(N_2) = \sum_{i=0}^{\infty} a_i N_2^i$ is absolutely convergent for $\mid N_2 \mid < r_0$ and $\Gamma(N_1(N_2), N_2) = 0$. That is, N_1 is an analytic function of N_2 when $\mid N_2 \mid < r_0$. ■

Proof of the Theorem 4.5: Since (40) and (41) are true if N_1 and N_2 satisfy the system. By Prop. 2 and Prop. 3, N_1 is an analytic function of N_2 for N_2 close to 0.
By the setting,

$$N_1(N_2) = a_0 + a_1 N_1 + a_2 N_2^2 + ...$$

we will show that $a_0 = 0$, $a_1 = 1$.
$a_0 = 0$ since $N_1(0) = 0$.
Because

$$\sum_{n=1}^{\infty} \frac{2n}{2n+1} N_1^{2n+1} = \sum_{n=1}^{\infty} \frac{2n(-1)^{n+1}}{2n+1} N_2^{2n+1}$$

or

$$\frac{2}{3}N_1^3 + \frac{4}{5}N_1^5 + ... = \frac{2}{3}N_2^3 - \frac{4}{5}N_2^5 + ...$$

Substitute N_1 by $\sum_{i=0}^{\infty} a_i N_2^i$ and compare the coefficient of N_2^3 on both sides:

$$\frac{2}{3}a_1^3 = \frac{2}{3}$$

then

$$a_1 = 1.$$

Comments: since $a_1 = 1$, and when $s \to 0$, $N_2 \to 0$,

$$\lim_{s\to 0} \frac{N_1}{N_2} = \lim_{N_2 \to 0} \frac{N_1}{N_2} = 1.$$

At the end of this part, we will use the analytic property of N_1 with respect to N_2 to show that both N_1 and N_2 are analytic functions of s.
Rewrite the differential equation corresponding to N_2':

$$N_2' = (1 + N_2^2)^2 \cdot \frac{N_1^2}{N_1^2 + N_2^2}$$

$$= (1 + N_2^2)^2 (1 - \frac{N_2^2}{N_1^2 + N_2^2})$$

$$= (1 + N_2^2)^2 (1 - \frac{1}{1 + (\frac{N_1}{N_2})^2})$$

Set$N_1 = \sum_{n=0}^{\infty} b_n N_2^n$, by the claim $\frac{N_1}{N_2}$ approaches to 1 as s get close to zero, and when $s \to 0$, N_2 also approaches to 0. so

$$\frac{N_1}{N_2} = \frac{b_0}{N_2} + b_1 + \sum_{n=2}^{\infty} b_n N_2^{n-1}$$

$$1 = \lim_{s\to 0} \frac{N_1}{N_2} = \lim_{s\to 0} (\frac{b_0}{N_2} + b_1)$$

thus, $b_0 = 0$, $b_1 = 1$ and $\frac{N_1}{N_2} = 1 + \sum_{n=2}^{\infty} b_n N_2^{n-1}$.
Set

$$(\frac{N_1}{N_2})^2 = 1 + \sum_{n=1}^{\infty} d_n N_2^n \equiv 1 + \phi(N_2),$$

where $\phi(N_2) = \sum_{n=1}^{\infty} d_n N_2^n$ is an analytic function of N_2 in a small neighborhood of 0.

$$\begin{aligned}\frac{1}{1+(\frac{N_1}{N_2})^2} &= \frac{1}{2+\phi(N_2)}\\ &= \frac{1}{2}\cdot\frac{1}{1+\frac{\phi(N_2)}{2}}\\ &= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(\frac{\phi(N_2)}{2})^n\end{aligned}$$

which is obviously an analytic function of N_2 in a neighborhood of 0 with radius of convergence 1.
Therefore, the right hand side of the above differential equation

$$(1+N_2^2)^2(1-\frac{1}{1+(\frac{N_1}{N_2})^2})$$

is also analytic with respect to N_2 in a small neighborhood of 0.
By Cauchy's theorem, $N_2' = (1+N_2^2)^2 \cdot \frac{N_1^2}{N_1^2+N_2^2}$, $N_2(0) = 0$ has a unique analytic solution $N_2 = N_2(s)$ in a small neighborhood of 0.
And because N_1 is an analytic function of N_2, then $N_1 = N_1(s)$ is also analytic when s is close to zero.
Because $\eta_i(s) = C^{-\frac{1}{2}} N_i(C^{\frac{1}{2}} s)$ for $i = 1, 2$, then η_1 and η_2 both are analytic in a neighborhood of 0.
Since $\xi_1 = \frac{F}{-1+C\eta_1^2}$ is a analytic function of η_1 in a neighborhood of 0 and η_1 is also analytic in a neighborhood of 0, then ξ_1 is analytic in a neighborhood of 0. And by the same argument, ξ_2 is also analytic in a neighborhood of 0. ∎

In fact, we can write down the first few terms of the power series solutions of N_1 and N_2 for the differential system:

$$N_1' = (1-N_1^2)^2 \cdot \frac{N_2^2}{N_1^2+N_2^2},$$
$$N_2' = (1+N_2^2)^2 \cdot \frac{N_1^2}{N_1^2+N_2^2},$$

with the initial conditions

$$N_1(0) = N_2(0) = 0.$$

By an easy calculation, we can get

$$N_1(s) = \frac{1}{2}s - \frac{1}{20}s^3 + \frac{1}{160}s^5 - \frac{29}{36000}s^7 + ...$$
$$N_2(s) = \frac{1}{2}s + \frac{1}{20}s^3 + \frac{1}{160}s^5 + \frac{29}{36000}s^7 + ...$$

5 the original system with all masses equal to 1

By the definition in section 3, the Hamiltonian

$$F = \frac{1}{\frac{m_1 m_2}{x_1} + \frac{m_3 m_4}{x_2}} \cdot (T - U - h)$$

$$= \frac{1}{\frac{m_1 m_2}{x_1} + \frac{m_3 m_4}{x_2}} (\frac{1}{2}[\frac{y_1^2}{m_1} + \frac{(y_1 - y_3)^2}{m_2} + \frac{(y_3 - y_2)^2}{m_3} + \frac{y_2^2}{m_4}]$$

$$-[\frac{m_1 m_2}{x_1} + \frac{m_1 m_3}{x_1 + x_3} + \frac{m_1 m_4}{x_1 + x_2 + x_3} + \frac{m_2 m_3}{x_3} + \frac{m_2 m_4}{x_2 + x_3} + \frac{m_3 m_4}{x_2}] - h)$$

For simplicity, assume $m_1 = m_2 = m_3 = m_4 = 1$, then

$$F = \frac{1}{\frac{1}{x_1} + \frac{1}{x_2}} (y_1^2 + y_2^2 + y_3^2 - y_1 y_3 - y_2 y_3)$$

$$-\frac{1}{\frac{1}{x_1} + \frac{1}{x_2}} (\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_1 + x_3} + \frac{1}{x_2 + x_3} + \frac{1}{x_1 + x_2 + x_3} + h)$$

$$= \frac{y_1^2 + y_2^2}{\frac{1}{x_1} + \frac{1}{x_2}} - \frac{y_1 + y_2}{\frac{1}{x_1} + \frac{1}{x_2}} \cdot y_3 - \frac{1}{\frac{1}{x_1} + \frac{1}{x_2}} [\frac{1}{x_3} + \frac{1}{x_1 + x_3} + \frac{1}{x_2 + x_3} + \frac{1}{x_1 + x_2 + x_3} + h - y_3^2] - 1$$

Right now we will do a canonical transformation such that $\frac{y_1 + y_2}{\frac{1}{x_1} + \frac{1}{x_2}} \cdot y_3$ can be absorbed into $\frac{y_1^2 + y_2^2}{\frac{1}{x_1} + \frac{1}{x_2}}$.

Let

$$Y_1 = y_1 - \frac{1}{2} y_3, \qquad Y_2 = y_2 - \frac{1}{2} y_3, \qquad Y_3 = y_3$$

and the generating function

$$W(x_1, x_2, x_3, Y_1, Y_2, Y_3) = x_1 (Y_1 + \frac{1}{2} Y_3) + x_2 (Y_2 + \frac{1}{2} Y_3) + x_3 Y_3,$$

satisfying

$$\frac{\partial W}{\partial x_i} = y_i, \qquad \text{and} \qquad \frac{\partial W}{\partial Y_i} = X_i$$

then

$$X_1 = \frac{\partial W}{\partial Y_1} = x_1$$

$$X_2 = \frac{\partial W}{\partial Y_2} = x_2$$

$$X_3 = \frac{\partial W}{\partial Y_3} = \frac{1}{2} x_1 + \frac{1}{2} x_2 + x_3$$

Under the above transformation, the new hamiltonian is

$$F = \frac{Y_1^2 + Y_2^2}{\frac{1}{X_1} + \frac{1}{X_2}} - \frac{1}{\frac{1}{X_1} + \frac{1}{X_2}} [\frac{1}{X_3 - \frac{1}{2} X_1 - \frac{1}{2} X_2}$$

$$+\frac{1}{X_3+\frac{1}{2}X_1-\frac{1}{2}X_2}+\frac{1}{X_3+\frac{1}{2}X_2-\frac{1}{2}X_1}+\frac{1}{X_3+\frac{1}{2}X_1+\frac{1}{2}X_2}+h-\frac{1}{2}Y_3^2]-1$$

Let

$$A=A(X_i,Y_3)$$
$$=[\frac{1}{X_3-\frac{1}{2}X_1-\frac{1}{2}X_2}+\frac{1}{X_3+\frac{1}{2}X_1-\frac{1}{2}X_2}+\frac{1}{X_3+\frac{1}{2}X_2-\frac{1}{2}X_1}+\frac{1}{X_3+\frac{1}{2}X_1+\frac{1}{2}X_2}+h-\frac{1}{2}Y_3^2],$$

then

$$F=\frac{Y_1^2+Y_2^2}{\frac{1}{X_1}+\frac{1}{X_2}}-\frac{1}{\frac{1}{X_1}+\frac{1}{X_2}}A(X_i,Y_3)-1$$

From the above Hamiltonian, we can get 6 differential equations:

$$X_1'=F_{Y_1}=\frac{2Y_1X_1X_2}{X_1+X_2}$$

$$X_2'=F_{Y_2}=\frac{2Y_2X_1X_2}{X_1+X_2}$$

$$X_3'=F_{Y_3}=\frac{Y_3X_1X_2}{X_1+X_2}$$

$$Y_1'=-F_{X_1}=-\frac{Y_1^2+Y_2^2}{(\frac{1}{X_1}+\frac{1}{X_2})^2}\cdot\frac{1}{X_1^2}+\frac{1}{(\frac{1}{X_1}+\frac{1}{X_2})^2}\cdot\frac{1}{X_1^2}A+\frac{1}{\frac{1}{X_1}+\frac{1}{X_2}}A_{X_1}$$
$$=(-F-1)\cdot\frac{1}{\frac{1}{X_1}+\frac{1}{X_2}}\cdot\frac{1}{X_1^2}+\frac{1}{\frac{1}{X_1}+\frac{1}{X_2}}A_{X_1}$$
$$Y_2'=-F_{X_2}=(-F-1)\cdot\frac{1}{\frac{1}{X_2}+\frac{1}{X_2}}\cdot\frac{1}{X_1^2}+\frac{1}{\frac{1}{X_1}+\frac{1}{X_2}}A_{X_2}$$
$$Y_3'=-F_{X_3}=\frac{A_{X_3}X_1X_2}{X_1+X_2}$$

Follow the idea of section 3(a),we have

$$\frac{dY_1}{dX_1}=\frac{Y_1'}{X_1'}=\frac{(-F-1)\cdot\frac{1}{\frac{1}{X_1}+\frac{1}{X_2}}\cdot\frac{1}{X_1^2}+\frac{1}{\frac{1}{X_1}+\frac{1}{X_2}}A_{X_1}}{\frac{2Y_1X_1X_2}{X_1+X_2}}$$

$$=\frac{\frac{-F-1}{X_1^2}+A_{X_1}}{2Y_1}$$

Therefore, by separating the variables, and integrating on the solution surface $F=0$,

$$\int 2Y_1dY_1=\int(\frac{-F-1}{X_1^2}+A_{X_1})dX_1$$

$$Y_1^2=\frac{F+1}{X_1}+\int A_{X_1}dX_1+C_1=\frac{1}{X_1}+\int A_{X_1}dX_1+C_1$$

Where C_1 is a constant with respect to X_1.

By a similar process,

$$Y_2^2 = \frac{F+1}{X_2} + \int A_{X_2} dX_2 + C_2 = \frac{1}{X_2} + \int A_{X_2} dX_2 + C_2$$

where C_2 is a constant with respect to X_2.

(a)**New transformation**

By the similar canonical transformation we defined in section 3(b):

$$\xi_1 = -X_1 Y_1^2, \quad \xi_2 = -X_2 Y_2^2, \quad \xi_3 = X_3, \quad \eta_1 = \frac{1}{Y_1}, \quad \eta_2 = \frac{1}{Y_2}, \quad \eta_3 = Y_3$$

$$X_1 = -\xi_1 \eta_1^2, \quad X_2 = -\xi_2 \eta_2^2, \quad X_3 = \xi_3, \quad Y_1 = \frac{1}{\eta_1}, \quad Y_2 = \frac{1}{\eta_2}, \quad Y_3 = \eta_3$$

therefore,

$$F = -\frac{\xi_1 \xi_2 (\eta_1^2 + \eta_2^2)}{\xi_1 \eta_1^2 + \xi_2 \eta_2^2} - 1$$

$$+\frac{\xi_1 \xi_2 \eta_1^2 \eta_2^2}{\xi_1 \eta_1^2 + \xi_2 \eta_2^2}[h - \frac{1}{2}\eta_3^2 + \frac{1}{-\frac{1}{2}\xi_1\eta_1^2 - \frac{1}{2}\xi_2\eta_2^2 + \xi_3} + \frac{1}{\frac{1}{2}\xi_1\eta_1^2 - \frac{1}{2}\xi_2\eta_2^2 + \xi_3}$$

$$+\frac{1}{-\frac{1}{2}\xi_1\eta_1^2 + \frac{1}{2}\xi_2\eta_2^2 + \xi_3} + \frac{1}{\frac{1}{2}\xi_1\eta_1^2 + \frac{1}{2}\xi_2\eta_2^2 + \xi_3}]$$

and the differential equations are

$$\xi_1' = \frac{2\xi_1\xi_2\eta_1\eta_2^2(\xi_1 - \xi_2)}{(\xi_1\eta_1^2 + \xi_2\eta_2^2)^2} + M_1$$

$$\xi_2' = \frac{-2\xi_1\xi_2\eta_2\eta_1^2(\xi_1 - \xi_2)}{(\xi_1\eta_1^2 + \xi_2\eta_2^2)^2} + M_2$$

$$\eta_1' = -F_{\xi_1} = \frac{\xi_2^2\eta_2^2(\eta_1^2 + \eta_2^2)}{(\xi_1\eta_1^2 + \xi_2\eta_2^2)^2} + G_1$$

$$\eta_2' = -F_{\xi_2} = \frac{\xi_1^2\eta_1^2(\eta_1^2 + \eta_2^2)}{(\xi_1\eta_1^2 + \xi_2\eta_2^2)^2} + G_2$$

$$\xi_3' = N_1$$

$$\eta_3' = N_2$$

Since we know, when $s \to 0$, ξ_3, η_3, ξ_1 and ξ_2 approach to nonzero constants. In the above equations, we can see M_1, M_2 , N_1 and N_2 are $O(s)$; G_1, G_2 are $O(s^2)$.

Introduce a new transformation

$$\frac{\xi_i+1}{s}=u_i, \qquad \frac{\eta_i}{s}-\frac{1}{2}=v_i, \quad i=1,2$$

$$\xi_3=\widehat{\xi}_3+u_3, \qquad \eta_3=\widehat{\eta}_3+v_3,$$

and

$$s=e^{-\tau}, \qquad ds=-sd\tau,$$

where $\widehat{\xi}_3$ and $\widehat{\eta}_3$ are the limits of ξ_3 and η_3 at $s=0$.
Then we can get a differential system about u_i and v_i:

$$\frac{du_1}{d\tau}=-F_{\eta_1}+u_1, \qquad \frac{dv_1}{d\tau}=F_{\xi_1}+v_1+\frac{1}{2},$$

$$\frac{du_2}{d\tau}=-F_{\eta_2}+u_2, \qquad \frac{dv_2}{d\tau}=F_{\xi_2}+v_2+\frac{1}{2},$$

$$\frac{du_3}{d\tau}=-e^{-\tau}F_{\eta_3}, \qquad \frac{dv_3}{d\tau}=e^{-\tau}F_{\xi_3},$$

and

$$\frac{ds}{d\tau}=-s.$$

(b)**Limits of u_i and v_i at $s=0(i=1,2)$**

Lemma 5.1

$$\lim_{s\to 0}u_1=\lim_{s\to 0}u_2=\lim_{s\to 0}v_1=\lim_{s\to 0}v_2=0$$

Proof According to the discussion in section 2, we know

$$\lim_{s\to 0}\frac{\eta_2^2}{\eta_1^2}=\lim_{t\to t_1}\frac{Y_1^2}{Y_2^2}=\lim_{t\to t_1}\frac{y_1^2}{y_2^2}=\lim_{t\to t_1}\frac{x_1p_1^2}{x_1p_4^2}=\frac{2(m_1m_2)^2}{(m_1+m_2)}\cdot\frac{(m_3+m_4)}{2\alpha(m_3m_4)^2}.$$

Since in our case $m_1=m_2=m_3=m_4$,

$$\lim_{s\to 0}\frac{\eta_2^2}{\eta_1^2}=1.$$

On the other hand, we know when t is very close to t_1, y_1 and y_2 are the same signs. Then $\lim_{s\to 0}\frac{\eta_2}{\eta_1}$ is positive. Therefore,

$$\lim_{s\to 0}\frac{\eta_2}{\eta_1}=1.$$

By L'Hospital rule,

$$\lim_{s\to 0}\frac{\eta_1}{s}=\lim_{s\to 0}\eta_1'=\lim_{s\to 0}\frac{\xi_2^2\eta_2^2(\eta_1^2+\eta_2^2)}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2}+\lim_{s\to 0}G_1$$

If the limit on the right hand side is finite, then the limit on the left hand side also exists and equals to the same value.

According to section 2, we have

$$\lim_{s\to 0}\eta_1 = \lim_{s\to 0}\eta_2 = 0,$$

and

$$\lim_{s\to 0}\xi_1 = \lim_{s\to 0}\xi_2 = -1.$$

So

$$\lim_{s\to 0}\frac{\xi_2^2\eta_2^2(\eta_1^2+\eta_2^2)}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2} = \lim_{s\to 0}\frac{\xi_2^2\frac{\eta_1^2}{\eta_2^2}(1+\frac{\eta_1^2}{\eta_2^2})}{(\xi_1+\xi_2\frac{\eta_1^2}{\eta_2^2})^2} = \frac{1\cdot 1\cdot 2}{(-1-1)^2} = \frac{1}{2}.$$

And it is obvious that

$$\lim_{s\to 0} G_1 = 0.$$

Therefore,

$$\lim_{s\to 0}\frac{\eta_1}{s} = \frac{1}{2},$$

and

$$\lim_{s\to 0}\frac{\eta_2}{s} = \lim_{s\to 0}\frac{\eta_2}{\eta_1}\cdot\lim_{s\to 0}\frac{\eta_1}{s} = \frac{1}{2}.$$

Here we can say

$$\lim_{s\to 0} v_1 = \lim_{s\to 0} v_2 = 0.$$

To consider the limit of u_i, we need to go back to X_i and Y_i.
Because $\lim_{s\to 0} X_1 Y_1^2 = 1$, and $Y_1^{-1} = \eta_1 = O(s)$, then $X_1 = O(s^2)$.
Since A is analytic and finite at $s=0$, A_{X_1} is also analytic at $s=0$. Consider the integral on the interval $[0, s_0]$, where s_0 is a small positive number. It is obvious that $X_1' = O(s)$ which is bounded on the interval $[0, s_0]$. And also A_{X_1} is also bounded because it is analytic at $s=0$. Hence $\int_0^{s_0} A_{X_1}X_1' ds = \lim_{\varepsilon\to 0}\int_\varepsilon^{s_0} A_{X_1}X_1' ds$ is bounded by some constant M_0.
On the other hand, we have

$$Y_1^2 = \frac{1}{X_1} + \int A_{X_1} dX_1 + C_1$$

$$X_1 Y_1^2 - 1 = X_1\int A_{X_1} dX_1 + C_1 X_1$$

that is

$$-(\xi_1+1) = X_1\int A_{X_1} dX_1 + C_1 X_1$$

$$-\frac{(\xi_1+1)}{s} = \frac{X_1}{s}\int A_{X_1} dX_1 + C_1\frac{X_1}{s}$$

Integrate on $[0, s_0]$, and we can see that

$$-\lim_{s\to 0}\frac{(\xi_1+1)}{s} = \lim_{s\to 0}\frac{X_1}{s}[\lim_{\varepsilon\to 0}\int_\varepsilon^{s_0} A_{X_1}X_1' ds + C_2]$$

Here constant C_2 depends on the choice of s_0. Then $\lim_{\varepsilon\to 0}\int_{\varepsilon}^{s_0} A_{X_1}X_1' ds + C_2 = \int_0^{s_0} A_{X_1}X_1' ds + C_2$ is bounded. Note that $X_1 = O(s^2)$, therefore,

$$\lim_{s\to 0}\frac{\xi_1+1}{s}=0, \qquad \text{or} \qquad \lim_{s\to 0} u_1 = 0.$$

From the equations, we can see

$$\lim_{s\to 0}\frac{\xi_2'}{\xi_1'} = -\lim_{s\to 0}\frac{\eta_1}{\eta_2} = -1,$$

and

$$\lim_{s\to 0}(\xi_1+1) = \lim_{s\to 0}(\xi_2+1) = 0,$$

by L'Hospital's rule,

$$\lim_{s\to 0}\frac{\xi_2+1}{\xi_1+1} = \lim_{s\to 0}\frac{\xi_2'}{\xi_1'} = -1.$$

Therefore,

$$\lim_{s\to 0} u_2 = \lim_{s\to 0}\frac{\xi_2+1}{s} = \lim_{s\to 0}\frac{\xi_2+1}{\xi_1+1}\cdot\lim_{s\to 0}\frac{\xi_1+1}{s} = (-1)\cdot 0 = 0. \qquad \blacksquare$$

(c)**Analytic solutions of u_i and v_i at $s = 0$**

So far, we've got a system of 6 differential equations with initial conditions $u_i(0) = v_i(0) = 0$.

$$s\frac{du_1}{ds} = F_{\eta_1} - u_1, \qquad s\frac{dv_1}{ds} = -F_{\xi_1} - v_1 - \frac{1}{2},$$
$$s\frac{du_2}{ds} = F_{\eta_2} - u_2, \qquad s\frac{dv_2}{ds} = -F_{\xi_2} - v_2 - \frac{1}{2},$$
$$\frac{du_3}{ds} = F_{\eta_3}, \qquad \frac{dv_3}{ds} = -F_{\xi_3}.$$

Let $s = e^{-\tau}$, this system can be rewritten as an autonomous system with seven variable u_i, v_i and s:

$$\frac{du_1}{d\tau} = -F_{\eta_1} + u_1, \qquad \frac{dv_1}{d\tau} = F_{\xi_1} + v_1 + \frac{1}{2},$$
$$\frac{du_2}{d\tau} = -F_{\eta_2} + u_2, \qquad \frac{dv_2}{d\tau} = F_{\xi_2} + v_2 + \frac{1}{2},$$
$$\frac{du_3}{d\tau} = -sF_{\eta_3}, \qquad \frac{dv_3}{d\tau} = sF_{\xi_3},$$

and

$$\frac{ds}{d\tau} = -s.$$

For simplification, we may use different notations:

$$\frac{d\sigma_k}{d\tau} = \Sigma_{l=1}^{7} b_{kl}\sigma_l + \varphi_k, \qquad (k = 1, ..., 7)$$

where $\sigma = (\sigma_1, ..., \sigma_7)^T = (u_1, u_2, v_1, v_2, u_3, v_3, s)^T$.
The initial value is $\sigma_k = 0(k{=}1,...,7)$ and φ_k are power series in σ_1, ...,σ_7 beginning with quadratic terms, and the b_{kl} are real constants.
The seven-by-seven matrix (b_{kl}) has the structure

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \omega \\ 1 & 0 & 0 & 0 & 0 & 0 & \omega \\ 0 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

where $\omega = \frac{1}{4}h - \frac{1}{8}\widehat{\eta}_3^2 + \frac{1}{\widehat{\xi}_3}$.

Theorem 5.2 *The system*

$$-s\frac{d\sigma}{ds} = B\sigma + \varphi, \qquad\qquad \varphi = (\varphi_1, ..., \varphi_7)^T$$

has the initial condition $\sigma = (\sigma_1, ..., \sigma_7)^T = 0$ *and*

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \omega \\ 1 & 0 & 0 & 0 & 0 & 0 & \omega \\ 0 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

where $\omega = \frac{1}{4}h - \frac{1}{8}\widehat{\eta}_3^2 + \frac{1}{\widehat{\xi}_3}$. *And also* φ_k*(k=1,2...,7) are power series in* σ_1, *...,*σ_7 *beginning with quadratic terms.*
Then this system has analytic solution σ *for* s *close to 0.*

It is easy to see that the eigenvalues of B are -1, -1,0, 0, 1, 1, 3 and B can be dialogized as

$$R = (r_{kl}) = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

So by a linear transformation

$$T = \begin{bmatrix} -1 & -\omega & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which satisfies $T^{-1}BT = R$,$\sigma = T(\rho_1, ..., \rho_7)^T \equiv T\rho$ and $(\chi_k) = T^{-1}(\varphi_k)$, the system can be changed to be

$$\frac{d\rho_k}{d\tau} = r_{kk}\rho_k + \chi_k, \qquad (k = 1, ..., 7)$$

where χ_k are also power series in ρ_1, ... ρ_7 beginning with quadratic terms.
Next, we will show that the above differential system

$$\frac{d\rho_k}{d\tau} = f_k(\rho) = r_{kk}\rho_k + \chi_k, \qquad (k = 1, ..., 7) \qquad (1)$$

has analytic solutions, where $\rho = (\rho_1, ..., \rho_7)$.
To find the solution, we will carry out substitutions of the special form

$$\mu_k = \rho_k - \phi_k(\rho_1, \rho_2) \qquad (k = 1, ..., 7) \qquad (2)$$

where ϕ_k are formal series in the first two variables ρ_1, ρ_2 only and begin with quadratic terms. If one sets

$$j_k(\mu) = j_k(\mu_1, ..., \mu_7) = \chi_k + r_{kk}\phi_k - \phi_{k\rho_1} f_1 - \phi_{k\rho_2} f_2, \qquad (k = 1, ..., 7)$$

where on the right ρ can be expressed as a function of μ by means of the substitution inverse to (2), then (1) becomes

$$\frac{d\mu_k}{d\tau} = r_{kk}\mu_k + j_k(\mu), \qquad (k = 1, ..., 7) \qquad (3)$$

with the power series j_k beginning again with quadratic terms. We will now determine the coefficients of the ϕ_k so that none of the series j_1, ..., j_7 contain product of powers of ρ_1, ρ_2 alone. In other words, the equations

$$j_k(\rho_1, \rho_2, 0, 0, 0, 0, 0) = 0, \qquad (k = 1, ..., 7) \qquad (4)$$

are to hold identically.
By (2) the ρ_1, ρ_2 are invertible power series in the two indeterminate variables μ_1, μ_2 only, and moreover, for $\mu_3 = 0, ..., \mu_7 = 0$ we have

$$\rho_k = \phi_k(\rho_1, \rho_2), \qquad (k = 3, ..., 7). \qquad (5)$$

Consequently, (4) reduces to the requirement that the equations

$$\chi_k + r_{kk}\phi_k - \phi_{k\rho_1} f_1 - \phi_{k\rho_2} f_2 = 0 \qquad (k = 1, ..., 7)$$

or

$$-r_{kk}\phi_k + \phi_{k\rho_1} r_{11}\rho_1 + \phi_{k\rho_2} r_{22}\rho_2 = \chi_k - \phi_{k\rho_1}\chi_1 - \phi_{k\rho_2}\chi_2, \qquad (k = 1, ..., 7) \tag{6}$$

be satisfied identically in ρ_1, ρ_2, where ρ_3,..., ρ_7 are defined by (5). Conversely, from (2), (5), (6) we again obtain (4). We now undertake comparison of coefficients in (6). If $\alpha\rho_1^{g_1}\rho_2^{g_2}$ is a term of ϕ_k with $g_1 + g_2 = m > 1$, the comparison gives

$$(-r_{kk} + g_1 r_{11} + g_2 r_{22})\alpha = \gamma$$

where γ is got from a polynomial in the coefficients of the terms in ϕ_1, ... ,ϕ_7 of degree less than m. Since $r_{11} = r_{22} = -1$ and $m > 1$,

$$-r_{kk} + g_1 r_{11} + g_2 r_{22} = -r_{kk} - m = 1 - m \neq 0, \qquad k = 1, 2$$

For $k = 3, ..., 7$, $r_{kk} \geq 0$, then of course $-r_{kk} - m \neq 0$. So

$$-r_{kk} + g_1 r_{11} + g_2 r_{22} = -r_{kk} - m \neq 0, \qquad (k = 1, ..., 7). \tag{7}$$

Therefore,induction shows that (4) has exactly one solution in power series ϕ_1, ... , ϕ_7. Next, we need to show the convergence of ϕ_k(k=1,...,7).

(d) **Method of majorants**

Convergence is proved by the method of majorants.

If

$$f = \sum_l a_{l_1...l_m} x_1^{l_1}...x_m^{l_m}, \qquad g = \sum_l b_{l_1...l_m} x_1^{l_1}...x_m^{l_m}$$

are two power series, which need not converge, then g is said to be a majorant of f, symbolically $f \prec g$, if

$$\mid a_{l_1...l_m} \mid \leq b_{l_1...l_m}$$

for all the coefficients.

Let

$$\rho_1 + \rho_2 + ... + \rho_7 = \Gamma, \qquad \chi_k \prec \frac{c_1\Gamma^2}{1 - c_1\Gamma}, \qquad (k = 1, ...7).$$

Since $r_{11} = r_{22} = -1$ and (7) is satisfied, it follows that

$$g_1 + g_2 < c_2 \mid -r_{kk} + g_1 r_{11} + g_2 r_{22} \mid \qquad (k = 1, ...7) \tag{8}$$

Consequently for the uniquely determined solution ψ_1, ... ,ψ_7 of

$$\psi_{k\rho_1}\rho_1 + \psi_{k\rho_2}\rho_2 = c_2(1 + \psi_{k\rho_1} + \psi_{k\rho_2})\frac{c_1\Gamma^2}{1 - c_1\Gamma} \qquad (k = 1, ...7) \tag{9}$$

$$\rho_k = \psi_k(\rho_1, \rho_2) \qquad (k = 3, ...7)$$

we have the relation $\phi_k \prec \psi_k$. The reason is following:
By the previous argument,we have

$$-r_{kk}\phi_k + \phi_{k\rho_1} r_{11}\rho_1 + \phi_{k\rho_2} r_{22}\rho_2 = \chi_k - \phi_{k\rho_1}\chi_1 - \phi_{k\rho_2}\chi_2, \qquad (k = 1, ..., 7)$$

If $\alpha\rho_1^{g_1}\rho_2^{g_2}$ is a term of ϕ_k with $g_1 + g_2 = m > 1$, the comparison gives

$$(-r_{kk} + g_1 r_{11} + g_2 r_{22})\alpha = \gamma$$

then

$$\mid \alpha \mid = \frac{\mid \gamma \mid}{\mid (-r_{kk} + g_1 r_{11} + g_2 r_{22}) \mid}$$

where γ comes from the right hand side of (6).
In the equations of ψ_k, if $\beta\rho_1^{g_1}\rho_2^{g_2}$ is a term of ϕ_k with $g_1 + g_2 = m > 1$, the comparison gives

$$(g_1 + g_2)\beta = c_2\gamma_1$$

where γ_1 comes from the right hand side of (9).
Since $\chi_k \prec \frac{c_1\Gamma^2}{1-c_1\Gamma}(k = 1, ...7)$, it is easy to see that $\gamma_1 > \mid \gamma \mid$. Then from (8),

$$\begin{aligned}\beta = \frac{c_2\gamma_1}{g_1 + g_2} &> \frac{c_2\gamma_1}{c_2 \mid -r_{kk} + g_1 r_{11} + g_2 r_{22} \mid} \\ &= \frac{\gamma_1}{\mid -r_{kk} + g_1 r_{11} + g_2 r_{22} \mid} \\ &> \frac{\mid \gamma \mid}{\mid -r_{kk} + g_1 r_{11} + g_2 r_{22} \mid} = \mid \alpha \mid . \blacksquare\end{aligned}$$

By (8), however, $\psi_1 = ... = \psi_7 = \psi$, and if in addition one sets $x_1 = x_2 = x$, it is evidently enough to prove the convergence for the solution $\psi(x)$ of

$$x\psi_x = (1 + \psi_x)\frac{c_3(x + \psi)^2}{1 - c_4(x + \psi)}.$$

On the other hand, let $\psi(x)/x = \overline{\psi}(x)$, then

$$(\overline{\psi} + x\overline{\psi}_x) = (1 + \overline{\psi} + x\overline{\psi}_x)\frac{c_3 x(1 + \overline{\psi})^2}{1 - c_4 x(1 + \overline{\psi})}$$

or

$$(\overline{\psi} + x\overline{\psi}_x)[1 - c_4 x(1 + \overline{\psi})] = (1 + \overline{\psi} + x\overline{\psi}_x)c_3 x(1 + \overline{\psi})^2$$

or

$$(\overline{\psi} + x\overline{\psi}_x) = c_4 x\overline{\psi} + c_4 x\overline{\psi}^2 + c_3 x(1 + \overline{\psi})^3 + x^2\overline{\psi}_x(1 + \overline{\psi})(c_4 + c_3 + c_3\overline{\psi}) \qquad (10)$$

Let$\overline{\psi} = \sum_{n=1}^{\infty} a_n x^n$, from (10) we can get the recursion formulas for a_k($k \geq 2$):

$$a_k(1 + k) = c_4 a_{k-1} + c_4 \sum_{m_1+m_2=k-1} a_{m_1}a_{m_2} + 3c_3 a_{k-1} + 3c_3 \sum_{m_1+m_2=k-1} a_{m_1}a_{m_2}$$

$$+c_3 \sum_{m_1+m_2+m_3=k-1} a_{m_1}a_{m_2}a_{m_3} + (c_4+c_3)(k-1)a_{k-1} + (c_4+2c_3) \sum_{m_1+m_2=k-1} m_1 a_{m_1}a_{m_2}$$

$$+c_3 \sum_{m_1+m_2+m_3=k-1} m_1 a_{m_1}a_{m_2}a_{m_3}$$

$$= [kc_4 + (k+2)c_3]a_{k-1} + (c_4+3c_3) \sum_{m_1+m_2=k-1} a_{m_1}a_{m_2} + (c_4+2c_3) \sum_{m_1+m_2=k-1} m_1 a_{m_1}a_{m_2}$$

$$+c_3 \sum_{m_1+m_2+m_3=k-1} a_{m_1}a_{m_2}a_{m_3} + c_3 \sum_{m_1+m_2+m_3=k-1} m_1 a_{m_1}a_{m_2}a_{m_3} \quad (11)$$

Consider the equation

$$\Psi[1 - c_4 x(1+\Psi)] = c_3 x(1+\Psi)^3$$

or

$$\Psi = c_4 x\Psi + c_4 x\Psi^2 + c_3 x(1+\Psi)^3 \quad (12)$$

Let $\Psi = \sum_{n=1}^{\infty} b_n x^n$, from (11) we can get some other recursion formulas for b_k($k \geq 2$):

$$b_k = c_4 b_{k-1} + c_4 \sum_{m_1+m_2=k-1} b_{m_1}b_{m_2} + 3c_3 b_{k-1} + 3c_3 \sum_{m_1+m_2=k-1} b_{m_1}b_{m_2}$$

$$+c_3 \sum_{m_1+m_2+m_3=k-1} b_{m_1}b_{m_2}b_{m_3}$$

$$= (c_4+3c_3)b_{k-1} + (c_4+3c_3) \sum_{m_1+m_2=k-1} b_{m_1}b_{m_2} + c_3 \sum_{m_1+m_2+m_3=k-1} b_{m_1}b_{m_2}b_{m_3}$$

Then

$$b_k(k+1) = (k+1)(c_4+3c_3)b_{k-1} + (k+1)(c_4+3c_3) \sum_{m_1+m_2=k-1} b_{m_1}b_{m_2}$$

$$+(k+1)c_3 \sum_{m_1+m_2+m_3=k-1} b_{m_1}b_{m_2}b_{m_3} \quad (13)$$

It is easy to see $a_1 = \frac{c_3}{2}$ and $b_1 = c_3$, so $a_1 < b_1$.
Assume $a_i < b_i$ for $i < k$, compare the terms on the right hand sides of (11) and (13):

$$(k+1)(c_4+3c_3)b_{k-1} > kc_4 + (k+2)c_3 a_{k-1}$$

$$(k+1)(c_4+3c_3) \sum_{m_1+m_2=k-1} b_{m_1}b_{m_2} > (c_4+3c_3) \sum_{m_1+m_2=k-1} a_{m_1}a_{m_2} +$$

$$(c_4+2c_3) \sum_{m_1+m_2=k-1} m_1 a_{m_1}a_{m_2}$$

$$(k+1)c_3 \sum_{m_1+m_2+m_3=k-1} b_{m_1}b_{m_2}b_{m_3} > c_3 \sum_{m_1+m_2+m_3=k-1} a_{m_1}a_{m_2}a_{m_3}$$

$$+c_3 \sum_{m_1+m_2+m_3=k-1} m_1 a_{m_1}a_{m_2}a_{m_3}$$

Therefore,

$$b_k(k+1) > a_k(k+1)$$

or

$$b_k > a_k.$$

By induction, $a_k < b_k$ is true for all k. So Ψ is a majorant of ψ.
From (12),we can see Ψ satisfies a cubic equation. Of course, it has convergent solution. Therefore, all the ϕ_k are convergent series. ■

By (3) and (4), one obtains for the given differential equation the particular solutions

$$\mu_k = \begin{cases} d_k e^{r_{kk}\tau}, & (k=1,2) \\ 0, \quad (k=3,...,7). \end{cases}$$

Since $r_{11} = r_{22} = -1$ and $s = e^{-\tau}$,

$$\mu_k = \begin{cases} d_1 e^{-\tau}, \\ e^{-\tau}, \\ 0, \quad (k=3,...,7) \end{cases}$$

where d_1 is an arbitrary constant.
Therefore, the solution for ρ_k is

$$\rho_k = \omega_k(\mu_1,\mu_2) = \omega_k(d_1 e^{-\tau}) = \omega_k(d_1 s) \qquad\qquad (k=1,...,7),$$

where ω_k are convergent power series in the variables μ_1 and μ_2 without a constant term, and d_1 is an arbitrary constant.
That is, ρ_k are convergent series of s in a sufficiently small neighborhood of $s=0$.
Then σ_k are also convergent power series of s in a sufficiently small neighborhood of $s=0$ for $k=1,...,7$.

6 Properties of the power series solutions

$$\xi_1 = -1 + u_1 s = -1 + ds^2 + (\frac{1}{15}d\omega - \frac{1}{5}d^2)s^4 + \\ (-\frac{1}{64}\omega^3 - \frac{23}{6300}d\omega^2 - \frac{26}{525}d^2\omega - \frac{1}{25}d^3)s^6 + O(s^8)$$

$$\xi_2 = -1 + u_2 s = -1 + (-\omega - d)s^2 + (-\frac{1}{5}d^2 - \frac{7}{15}d\omega - \frac{4}{15}\omega^2)s^4 + \\ (-\frac{2167}{100800}\omega^3 + \frac{31}{1260}d\omega^2 + \frac{37}{525}d^2\omega + \frac{1}{25}d^3)s^6 + O(s^8)$$

$$\eta_1 = \frac{s}{2} + v_1 s = \frac{s}{2} + \frac{3d + \frac{1}{4}\omega}{15}s^3 + (\frac{1}{10}d^2 + \frac{1}{70}d\omega - \frac{1}{840}\omega^2)s^5 +$$

$$(-\frac{1093}{336000}\omega^3 - \frac{1}{2625}d\omega^2 + \frac{39}{3500}d^2\omega + \frac{58}{1125}d^3)s^7 + O(s^9)$$

$$\eta_2 = \frac{s}{2} + v_2 s = \frac{s}{2} + \frac{-\frac{11}{4}\omega - 3d}{15}s^3 + (\frac{1}{10}d^2 + \frac{13}{70}d\omega + \frac{71}{840}\omega^2)s^5 +$$

$$(-\frac{6233}{144000}\omega^3 - \frac{33}{250}d\omega^2 - \frac{1507}{10500}d^2\omega - \frac{58}{1125}d^3)s^7 + O(s^9)$$

$$\xi_3 = \widehat{\xi}_3 + u_3 = \widehat{\xi}_3 + \frac{1}{24}\widehat{\eta}_3 s^3 + \frac{1}{240}\omega\widehat{\eta}_3 s^5 - \frac{1}{288\widehat{\xi}_3^2}s^6 + \frac{\widehat{\eta}_3}{7}(-\frac{1}{900}\omega^2 - \frac{1}{100}d\omega - \frac{1}{100}d^2)s^7 + ...$$

$$\eta_3 = \widehat{\eta}_3 + v_3 = \widehat{\eta}_3 - \frac{1}{6\widehat{\xi}_3^2}s^3 - \frac{\omega}{60\widehat{\xi}_3^2}s^5 + \frac{\widehat{\eta}_3}{144\widehat{\xi}_3^3}s^6 + \frac{1}{7\widehat{\xi}_3^2}(-\frac{611}{14400}\omega^2 + \frac{1}{25}d\omega + \frac{1}{25}d^2)s^7...$$

where

$$\omega = \frac{1}{4}h - \frac{1}{8}\widehat{\eta}_3^2 + \frac{1}{\widehat{\xi}_3} = \frac{1}{4}\lim_{s\to 0} A,$$

and d is an arbitrary constant.
If we set $\widetilde{d} = d + \frac{\omega}{2}$, then the first four power series solutions can be rewritten as:

$$\xi_1 = -1 + (\widetilde{d} - \frac{\omega}{2})s^2 + (-\frac{1}{5}\widetilde{d}^2 + \frac{4}{15}\widetilde{d}\omega - \frac{1}{12}\omega^2)s^4 +$$

$$(-\frac{61}{2880}\omega^3 + \frac{1}{63}\widetilde{d}\omega^2 + \frac{11}{1050}\widetilde{d}^2\omega - \frac{1}{25}\widetilde{d}^3)s^6 + O(s^8)$$

$$\xi_2 = -1 + (-\widetilde{d} - \frac{\omega}{2})s^2 + (-\frac{1}{5}\widetilde{d}^2 - \frac{4}{15}\widetilde{d}\omega - \frac{1}{12}\omega^2)s^4 +$$

$$(-\frac{61}{2880}\omega^3 - \frac{1}{63}\widetilde{d}\omega^2 + \frac{11}{1050}\widetilde{d}^2\omega + \frac{1}{25}\widetilde{d}^3)s^6 + O(s^8)$$

$$\eta_1 = \frac{s}{2} + (\frac{1}{5}\widetilde{d} - \frac{\omega}{12})s^3 + (\frac{1}{10}\widetilde{d}^2 - \frac{3}{35}\widetilde{d}\omega + \frac{1}{60}\omega^2)s^5 +$$

$$(-\frac{6775}{1008000}\omega^3 + \frac{19}{700}\widetilde{d}\omega^2 - \frac{139}{2100}\widetilde{d}^2\omega + \frac{58}{1125}\widetilde{d}^3)s^7 + O(s^9)$$

$$\eta_2 = \frac{s}{2} + (-\frac{1}{5}\widetilde{d} - \frac{\omega}{12})s^3 + (\frac{1}{10}\widetilde{d}^2 + \frac{3}{35}\widetilde{d}\omega + \frac{1}{60}\omega^2)s^5 +$$

$$(-\frac{6775}{1008000}\omega^3 - \frac{19}{700}\widetilde{d}\omega^2 - \frac{139}{2100}\widetilde{d}^2\omega - \frac{58}{1125}\widetilde{d}^3)s^7 + O(s^9)$$

It tells us that each pair $\xi_1 = \xi_2$, $\eta_1 = \eta_2$ when $\widetilde{d} = 0$ or $d = -\frac{\omega}{2}$.
In fact, by the symmetry of the equation, if $(\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3)$ is a solution of the differential system, then $(\xi_2, \xi_1, \xi_3, \eta_2, \eta_1, \eta_3)$ is also a solution of the same system.

Basically, in the solutions, the coefficients of ξ_3 and η_3 has nothing to do with d up to the power s^6. And ξ_1, ξ_2, η_1, η_2 has no mixed term $d\omega$ for the first two nonzero terms in the

power series solutions.
Compare this solution with the solution for the decoupled case,

$$\xi_1^0 = F(-1 - \frac{1}{4}Cs^2 - \frac{1}{80}C^2s^4 + \frac{1}{1600}C^3s^6 + \frac{7}{288000}C^4s^8) + ...$$

$$\xi_2^0 = F(-1 + \frac{1}{4}Cs^2 - \frac{1}{80}C^2s^4 - \frac{1}{1600}C^3s^6 + \frac{7}{288000}C^4s^8) + ...$$

$$\eta_1^0 = \frac{1}{2}s - \frac{C}{20}s^3 + \frac{C^2}{160}s^5 - \frac{29C^3}{36000}s^7 + ...$$

$$\eta_2^0 = \frac{1}{2}s + \frac{C}{20}s^3 + \frac{C^2}{160}s^5 + \frac{29C^3}{36000}s^7 + ...$$

where η_1^0 and η_2^0 are odd functions of s, and $\eta_1^0(C) = \eta_2^0(-C)$;
ξ_1^0 and ξ_2^0 are even functions of s, and $\xi_1^0(C) = \xi_2^0(-C)$;
and F approaches 1 as s approaches 0, which means that $F = 1$ on the phase space of the solutions for the decoupled case.

If we let $C = -4\widetilde{d}$, then

$$\xi_1^0 = -1 + \widetilde{d}s^2 - \frac{1}{5}\widetilde{d}^2s^4 - \frac{1}{25}\widetilde{d}^3s^6 + \frac{7}{1125}\widetilde{d}^4s^8 + ...$$

$$\xi_2^0 = -1 - \widetilde{d}s^2 - \frac{1}{5}\widetilde{d}^2s^4 + \frac{1}{25}\widetilde{d}^3s^6 + \frac{7}{1125}\widetilde{d}^4s^8 + ...$$

$$\eta_1^0 = \frac{1}{2}s + \frac{\widetilde{d}}{5}s^3 + \frac{\widetilde{d}^2}{10}s^5 + \frac{58\widetilde{d}^3}{1125}s^7 + ...$$

$$\eta_2^0 = \frac{1}{2}s - \frac{\widetilde{d}}{5}s^3 + \frac{\widetilde{d}^2}{10}s^5 - \frac{58\widetilde{d}^3}{1125}s^7 + ...$$

Therefore,

$$\xi_1 = \xi_1^0 - \frac{\omega}{2}s^2 + O(s^4)$$

$$\xi_2 = \xi_2^0 - \frac{\omega}{2}s^2 + O(s^4)$$

$$\eta_1 = \eta_1^0 - \frac{1}{12}\omega s^3 + O(s^5)$$

$$\eta_2 = \eta_2^0 - \frac{1}{12}\omega s^3 + O(s^5)$$

The above results tell us that
1. In each of the decoupled case and the coupled case, it has one parameter $\widetilde{d}$, which is an arbitrary constant. From the comparison, we can see those two constants are the same. Recall the meaning of C in section 3,

$$\widetilde{d} = -\frac{C}{4} = -\frac{1}{4}\lim_{s\to 0}\frac{\xi_1 - \xi_2}{\xi_1\eta_1^2 + \xi_2\eta_2^2};$$

2.The motion of the decoupled case and the coupled case are very similar. Up to the power s^4, the coupled case can be considered as a decoupled case adding another motion

which is related to the initial conditions: h, $\widehat{\xi}_3$ and $\widehat{\eta}_3$;

3. Because of the mixed term $d\omega$, the coupled solution can NOT be considered exactly as the sum of a decoupled solution and a special solution which has nothing to do with d;

4. Up to the power s^7, the solution ξ_1, ξ_2, η_1, and η_2 are still symmetric with respect to the new constant $\widetilde{d}$: $\xi_1(\widetilde{d}) = \xi_2(-\widetilde{d})$ and $\eta_1(\widetilde{d}) = \eta_2(-\widetilde{d})$;

5. In the solution of the coupled case, basically there are two constants: $\widetilde{d} = -\frac{1}{4}C$, where C is the constant in the decoupled case; another one ω is given by the initial conditions and it shows the effect of the coupling terms to the solutions, and also the coupling term $\widetilde{d}\omega$ will start appearing from the term s^6 in the power series form of X_1 and X_2;

6. Since ω is fixed for given initial values, but $\widetilde{d}$ will make the solution to be a one-parameter family which is similar to the decoupled case, and the analytic solution can ONLY happen if we choose the same common constant $\widetilde{d}$ on both negative and positive sides of s.

7 Change of Variables for the system with general masses

By the definition in section 3, the Hamiltonian

$$F = \frac{1}{\frac{m_1 m_2}{x_1} + \frac{m_3 m_4}{x_2}} \cdot (T - U - h)$$

$$= \frac{1}{\frac{m_1 m_2}{x_1} + \frac{m_3 m_4}{x_2}} (\frac{1}{2}[\frac{y_1^2}{m1} + \frac{(y_1 - y_3)^2}{m_2} + \frac{(y_3 - y_2)^2}{m_3} + \frac{y_2^2}{m_4}]$$

$$-[\frac{m_1 m_2}{x_1} + \frac{m_1 m_3}{x_1 + x_3} + \frac{m_1 m_4}{x_1 + x_2 + x_3} + \frac{m_2 m_3}{x_3} + \frac{m_2 m_4}{x_2 + x_3} + \frac{m_3 m_4}{x_2}] - h)$$

$$= \frac{1}{\frac{m_1 m_2}{x_1} + \frac{m_3 m_4}{x_2}} \{(\frac{1}{2m_1} + \frac{1}{2m_2}) y_1^2 + (\frac{1}{2m_3} + \frac{1}{2m_4}) y_2^2 - (\frac{y_1}{m_2} + \frac{y_2}{m_3}) y_3$$

$$+(\frac{1}{2m_2} + \frac{1}{2m_3}) y_3^2 - [\frac{m_1 m_2}{x_1} + \frac{m_1 m_3}{x_1 + x_3} + \frac{m_1 m_4}{x_1 + x_2 + x_3} + \frac{m_2 m_3}{x_3} + \frac{m_2 m_4}{x_2 + x_3} + \frac{m_3 m_4}{x_2}] - h\}$$

Let

$$Y_1 = y_1 - \frac{m_1}{m_1 + m_2} y_3, \qquad Y_2 = y_2 - \frac{m_4}{m_3 + m_4} y_3, \qquad Y_3 = y_3$$

Let the generating function is

$$W(x_i, Y_i) = x_1 (Y_1 + \frac{m_1}{m_1 + m_2} Y_3) + x_2 (Y_2 + \frac{m_4}{m_3 + m_4} Y_3) + x_3 Y_3$$

So

$$X_1 = W_{Y_1} = x_1, \qquad X_2 = W_{Y_2} = x_2, \qquad X_3 = W_{Y_3} = \frac{m_1}{m_1+m_2}x_1 + \frac{m_4}{m_3+m_4}x_2 + x_3.$$

And the new hamiltonian is

$$F = \frac{1}{\frac{m_1m_2}{X_1} + \frac{m_3m_4}{X_2}}\{\frac{m_1+m_2}{2m_1m_2}Y_1^2 + \frac{m_3+m_4}{2m_3m_4}Y_2^2$$

$$+[\frac{m_2+m_3}{2m_2m_3} - \frac{m_1}{2m_2(m_1+m_2)} - \frac{m_4}{2m_3(m_3+m_4)}]Y_3^2$$

$$-[\frac{m_1m_2}{X_1} + \frac{m_1m_3}{X_3 + \frac{m_2}{m_1+m_2}X_1 - \frac{m_4}{m_3+m_4}X_2} + \frac{m_1m_4}{X_3 + \frac{m_2}{m_1+m_2}X_1 + \frac{m_3}{m_3+m_4}X_2}$$

$$+\frac{m_2m_3}{X_3 - \frac{m_1}{m_1+m_2}X_1 - \frac{m_4}{m_3+m_4}X_2} + \frac{m_2m_4}{X_3 - \frac{m_1}{m_1+m_2}X_1 + \frac{m_3}{m_3+m_4}X_2} + \frac{m_3m_4}{X_2}] - h\}$$

$$= \frac{1}{\frac{m_1m_2}{X_1} + \frac{m_3m_4}{X_2}}(\frac{m_1+m_2}{2m_1m_2}Y_1^2 + \frac{m_3+m_4}{2m_3m_4}Y_2^2)$$

$$+\frac{1}{\frac{m_1m_2}{X_1} + \frac{m_3m_4}{X_2}}\{[\frac{m_2+m_3}{2m_2m_3} - \frac{m_1}{2m_2(m_1+m_2)} - \frac{m_4}{2m_3(m_3+m_4)}]Y_3^2$$

$$-[\frac{m_1m_3}{X_3 + \frac{m_2}{m_1+m_2}X_1 - \frac{m_4}{m_3+m_4}X_2} + \frac{m_1m_4}{X_3 + \frac{m_2}{m_1+m_2}X_1 + \frac{m_3}{m_3+m_4}X_2}$$

$$+\frac{m_2m_3}{X_3 - \frac{m_1}{m_1+m_2}X_1 - \frac{m_4}{m_3+m_4}X_2} + \frac{m_2m_4}{X_3 - \frac{m_1}{m_1+m_2}X_1 + \frac{m_3}{m_3+m_4}X_2}] - h\} - 1$$

Follow the similar canonical transformation in section 3(b):

$$\xi_1 = -X_1Y_1^2, \quad \xi_2 = -X_1Y_1^2, \quad \xi_3 = X_3, \quad \eta_1 = \frac{1}{Y_1}, \quad \eta_2 = \frac{1}{Y_2}, \quad \eta_3 = Y_3$$

$$X_1 = -\xi_1\eta_1^2, \quad X_2 = -\xi_1\eta_1^2, \quad X_3 = \xi_3, \quad Y_1 = \frac{1}{\eta_1}, \quad Y_2 = \frac{1}{\eta_2}, \quad Y_3 = \eta_3$$

and the new Hamiltonian is

$$F = -\frac{\xi_1\xi_2(\frac{m_1+m_2}{2m_1m_2}\eta_2^2 + \frac{m_3+m_4}{2m_3m_4}\eta_1^2)}{m_3m_4\xi_1\eta_1^2 + m_1m_2\xi_2\eta_2^2}$$

$$+\frac{\xi_1\xi_2\eta_1^2\eta_2^2}{m_3m_4\xi_1\eta_1^2 + m_1m_2\xi_2\eta_2^2}\{[-\frac{m_2+m_3}{2m_2m_3} + \frac{m_1}{2m_2(m_1+m_2)} + \frac{m_4}{2m_3(m_3+m_4)}]\eta_3^2 + h$$

$$+\frac{m_1m_3}{\xi_3 - \frac{m_2}{m_1+m_2}\xi_1\eta_1^2 + \frac{m_4}{m_3+m_4}\xi_2\eta_2^2} + \frac{m_1m_4}{\xi_3 - \frac{m_2}{m_1+m_2}\xi_1\eta_1^2 - \frac{m_3}{m_3+m_4}\xi_2\eta_2^2} + \frac{m_2m_3}{\xi_3 + \frac{m_1}{m_1+m_2}\xi_1\eta_1^2 + \frac{m_4}{m_3+m_4}\xi_2\eta_2^2}$$

$$+\frac{m_2m_4}{\xi_3 + \frac{m_1}{m_1+m_2}\xi_1\eta_1^2 - \frac{m_3}{m_3+m_4}\xi_2\eta_2^2}\} - 1$$

The equations corresponding to ξ_1, ξ_2, η_1 and η_2 are

$$\xi_1' = F_{\eta_1} = \frac{2\xi_1\xi_2\eta_1\eta_2^2(\frac{m_3m_4(m_1+m_2)}{2m_1m_2}\xi_1 - \frac{m_1m_2(m_3+m_4)}{2m_3m_4}\xi_2)}{(m_3m_4\xi_1\eta_1^2 + m_1m_2\xi_2\eta_2^2)^2} + ...$$

$$\xi_2' = F_{\eta_2} = \frac{-2\xi_1\xi_2\eta_1^2\eta_2(\frac{m_3m_4(m_1+m_2)}{2m_1m_2}\xi_1 - \frac{m_1m_2(m_3+m_4)}{2m_3m_4}\xi_2)}{(m_3m_4\xi_1\eta_1^2 + m_1m_2\xi_2\eta_2^2)^2} + ...$$

$$\eta_1' = -F_{\xi_1} = \frac{m_1m_2\xi_2^2\eta_2^2(\frac{m_1+m_2}{2m_1m_2}\eta_2^2 + \frac{m_3+m_4}{2m_3m_4}\eta_1^2)}{(m_3m_4\xi_1\eta_1^2 + m_1m_2\xi_2\eta_2^2)^2} + ...$$

$$\eta_2' = -F_{\xi_2} = \frac{m_3m_4\xi_1^2\eta_1^2(\frac{m_1+m_2}{2m_1m_2}\eta_2^2 + \frac{m_3+m_4}{2m_3m_4}\eta_1^2)}{(m_3m_4\xi_1\eta_1^2 + m_1m_2\xi_2\eta_2^2)^2} + ...$$

Consider the limit of ξ_i and η_i at $s = 0$: Similar to the previous argument, we can see

$$\lim_{s\to 0} \frac{\eta_2^2}{\eta_1^2} = \frac{2m_1^2m_2^2}{m_1+m_2} \cdot \frac{m_3+m_4}{2m_3^2m_4^2} \cdot (\frac{m_3+m_4}{m_1+m_2})^{\frac{1}{3}},$$

$$\lim_{s\to 0} \xi_1 = -\frac{2m_1^2m_2^2}{m_1+m_2},$$

$$\lim_{s\to 0} \xi_2 = -\frac{2m_3^2m_4^2}{m_3+m_4},$$

$$\lim_{s\to 0} \eta_1 = \lim_{s\to 0} \eta_2 = 0,$$

$$\lim_{s\to 0} \frac{\xi_1 + \frac{2m_1^2m_2^2}{m_1+m_2}}{s} = \lim_{s\to 0} \xi_1' = 0, \qquad \lim_{s\to 0} \frac{\xi_2 + \frac{2m_3^2m_4^2}{m_3+m_4}}{s} = \lim_{s\to 0} \xi_2' = 0,$$

$$\lim_{s\to 0} \frac{\eta_1}{s} = \lim_{s\to 0} \eta_1' = \frac{(m_1+m_2)(m_3+m_4)^{\frac{1}{3}}}{2m_1m_2[m_1m_2(m_3+m_4)^{\frac{1}{3}} + m_3m_4(m_1+m_2)^{\frac{1}{3}}]},$$

$$\lim_{s\to 0} \frac{\eta_1}{s} = \lim_{s\to 0} \eta_1' = \frac{(m_1+m_2)^{\frac{1}{3}}(m_3+m_4)}{2m_3m_4[m_1m_2(m_3+m_4)^{\frac{1}{3}} + m_3m_4(m_1+m_2)^{\frac{1}{3}}]}.$$

Denote $\lim_{s\to 0} \frac{\eta_1}{s} = \widehat{v}_1$ and $\lim_{s\to 0} \frac{\eta_2}{s} = \widehat{v}_2$. Do the change of variable

$$u_1 = \frac{\xi_1 + \frac{2m_1^2m_2^2}{m_1+m_2}}{s}, \qquad u_2 = \frac{\xi_2 + \frac{2m_3^2m_4^2}{m_3+m_4}}{s},$$

$$v_1 = \frac{\eta_1}{s} - \widehat{v}_1, \qquad v_2 = \frac{\eta_2}{s} - \widehat{v}_2,$$

$$u_3 = \xi_3 - \widehat{\xi}_3, \qquad v_3 = \eta_3 - \widehat{\eta}_3$$

where $\widehat{\xi}_3$ and $\widehat{\eta}_3$ are the limits of ξ_3 and η_3 at $s = 0$.
Then the new equations become:

$$su_1' = F_{\eta_1} - u_1, \qquad su_2' = F_{\eta_2} - u_2,$$

$$sv_1' = -F_{\xi_1} - v_1 - \widehat{v}_1, \qquad sv_2' = -F_{\xi_2} - v_2 - \widehat{v}_2,$$

$$u_3' = F_{\eta_3}, \qquad v_3' = -F_{\xi_3},$$

with the initial conditions at $s = 0$:

$$u_i(0) = v_i(0) = 0 \qquad (i = 1, 2, 3)$$

Let $s = e^{-\tau}$, the above equations can be rewritten as an autonomous system:

$$\frac{du_1}{d\tau} = -F_{\eta_1} + u_1, \qquad \frac{du_2}{d\tau} = -F_{\eta_2} + u_2,$$

$$\frac{dv_1}{d\tau} = F_{\xi_1} + v_1 + \widehat{v}_1, \qquad \frac{dv_2}{d\tau} = F_{\xi_2} + v_2 + \widehat{v}_2,$$

$$\frac{du_3}{d\tau} = -sF_{\eta_3}, \qquad v_3' = sF_{\xi_3},$$

and

$$\frac{ds}{d\tau} = -s.$$

For simplification, we may use different notations:

$$\frac{d\sigma_k}{d\tau} = \Sigma_{l=1}^{7} b_{kl}\sigma_l + \varphi_k, \qquad (k = 1, ..., 7) \qquad (8.1)$$

The initial value is $\sigma_k = 0(k$=1,...,7) and φ_k are power series in σ_1, ...,σ_7 beginning with quadratic terms, and the b_{kl} are real constants.
The seven-by-seven matrix (b_{kl}) has the structure

$$B = \begin{bmatrix} b_{11} & b_{12} & 0 & 0 & 0 & 0 & b_{17} \\ b_{21} & b_{22} & 0 & 0 & 0 & 0 & b_{27} \\ 0 & 0 & b_{33} & b_{34} & 0 & 0 & 0 \\ 0 & 0 & b_{43} & b_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

where

$$b_{11} = 1 - \frac{2m_3m_4\sqrt[3]{m_1+m_2}}{m_1m_2\sqrt[3]{m_3+m_4} + m_3m_4\sqrt[3]{m_1+m_2}};$$

$$b_{12} = \frac{2m_1^2m_2^2(m_3+m_4)}{m_3m_4(m_1+m_2)^{\frac{2}{3}}} \cdot \frac{1}{m_1m_2\sqrt[3]{m_3+m_4} + m_3m_4\sqrt[3]{m_1+m_2}};$$

$$b_{21} = \frac{2m_3^2m_4^2(m_1+m_2)}{m_1m_2(m_3+m_4)^{\frac{2}{3}}} \cdot \frac{1}{m_1m_2\sqrt[3]{m_3+m_4} + m_3m_4\sqrt[3]{m_1+m_2}};$$

$$b_{22} = 1 - \frac{2m_1m_2\sqrt[3]{m_3+m_4}}{m_1m_2\sqrt[3]{m_3+m_4}+m_3m_4\sqrt[3]{m_1+m_2}};$$

$$b_{33} = 1 + \frac{2m_3m_4\sqrt[3]{m_1+m_2}}{m_1m_2\sqrt[3]{m_3+m_4}+m_3m_4\sqrt[3]{m_1+m_2}};$$

$$b_{34} = \frac{-2m_3^2m_4^2(m_1+m_2)}{m_1m_2(m_3+m_4)^{\frac{2}{3}}} \cdot \frac{1}{m_1m_2\sqrt[3]{m_3+m_4}+m_3m_4\sqrt[3]{m_1+m_2}};$$

$$b_{43} = \frac{-2m_1^2m_2^2(m_3+m_4)}{m_3m_4(m_1+m_2)^{\frac{2}{3}}} \cdot \frac{1}{m_1m_2\sqrt[3]{m_3+m_4}+m_3m_4\sqrt[3]{m_1+m_2}};$$

$$b_{44} = 1 + \frac{2m_1m_2\sqrt[3]{m_3+m_4}}{m_1m_2\sqrt[3]{m_3+m_4}+m_3m_4\sqrt[3]{m_1+m_2}};$$

$$b_{17} = [h - \frac{m_1+m_2+m_3+m_4}{2m_2m_3(m_1+m_2)(m_3+m_4)}\widehat{\eta}_3^2 + (m_1m_3+m_2m_3+m_1m_4+m_2m_4)\frac{1}{\widehat{\xi}_3}]$$

$$\cdot\frac{2m_1^2m_2^2(m_3+m_4)}{(m_1m_2\sqrt[3]{m_3+m_4}+m_3m_4\sqrt[3]{m_1+m_2})^3};$$

$$b_{27} = [h - \frac{m_1+m_2+m_3+m_4}{2m_2m_3(m_1+m_2)(m_3+m_4)}\widehat{\eta}_3^2 + (m_1m_3+m_2m_3+m_1m_4+m_2m_4)\frac{1}{\widehat{\xi}_3}]$$

$$\cdot\frac{2m_3^2m_4^2(m_1+m_2)}{(m_1m_2\sqrt[3]{m_3+m_4}+m_3m_4\sqrt[3]{m_1+m_2})^3};$$

To find the eigenvalues of B, we only need to find the eigenvalues for the two different 2 by 2 matrices:

$$B_1 = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \qquad B_2 = \begin{bmatrix} b_{33} & b_{34} \\ b_{43} & b_{44} \end{bmatrix}$$

By carefully calculation, we find out that the eigenvalues for B_1 are 1 and -1; the eigenvalues for B_2 are 1 and 3. Fortunately, they are exactly the same as the case with equal masses. And after some calculation, we can see that B is similar to the same diagonal matrix R:

$$R = (r_{kl}) = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore, the previous argument works. And we will also have the analytic properties of the solutions of u_i and v_i at a neighborhood of $s = 0$.

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